

# EMBEDDING $\ell_\infty$ INTO THE SPACE OF BOUNDED OPERATORS ON CERTAIN BANACH SPACES

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ABSTRACT. Sufficient conditions are given on a Banach space  $X$  which ensure that  $\ell_\infty$  embeds in  $\mathcal{L}(X)$ , the space of all bounded linear operators on  $X$ . A basic sequence  $(e_n)$  is said to be quasisubsymmetric if for any two increasing sequences  $(k_n)$  and  $(\ell_n)$  of positive integers with  $k_n \leq \ell_n$  for all  $n$ ,  $(e_{k_n})$  dominates  $(e_{\ell_n})$ . If a Banach space  $X$  has a seminormalized quasisubsymmetric basis then  $\ell_\infty$  embeds in  $\mathcal{L}(X)$ .

## 1. INTRODUCTION

The famous open problem of whether there exists an infinite-dimensional Banach space on which every (bounded linear) operator is a compact perturbation of a multiple of the identity, is attributed to S. Banach (see related papers [G], [GM], and [S]). One of the reasons that this problem has attracted a lot of attention is that if such a space  $X$  exists then by the results of [AS] or [L],  $X$  provides a positive solution to the Invariant Subspace Problem for Banach spaces, namely every operator on  $X$  has a non-trivial (i.e. different from zero and the whole space) invariant subspace. Notice that if a space  $X$  satisfies the assumptions of the above problem of Banach and if in addition  $X$  has the *Approximation Property* (AP) and separable dual then  $\mathcal{L}(X)$ , the space of all operators on  $X$  endowed with the usual operator norm, is separable (recall that a Banach space  $X$  has the AP if for every compact subset  $K$  of  $X$  and every  $\varepsilon > 0$  there exists a finite rank operator  $S \in \mathcal{L}(X)$  such that  $\|x - Sx\| \leq \varepsilon$  for all  $x \in K$ ). Thus if a reflexive Banach space  $X$  with a basis satisfies the assumptions of the above problem of Banach then  $\mathcal{L}(X)$  is separable (recall that a sequence  $(e_n)_n$  of non-zero elements of a Banach space is a *basic sequence* if there exists a constant  $B$  such that  $\|\sum_{n=1}^N a_n e_n\| \leq B \|\sum_{n=1}^M a_n e_n\|$  for all integers  $N \leq M$  and all sequences of scalars  $(a_n)$ ; the smallest such constant  $B$  is called the *basis constant* of  $(e_n)$ ; the sequence  $(e_n)_n$  is called a *basis* for  $X$  if it is a basic sequence whose linear span is dense in  $X$ ). In this paper we provide sufficient conditions on a Banach space  $X$  which imply that  $\ell_\infty$  embeds in  $\mathcal{L}(X)$  (which we denote by “ $\ell_\infty \hookrightarrow \mathcal{L}(X)$ ”) and hence that  $\mathcal{L}(X)$  is non-separable. The question whether  $\mathcal{L}(X)$  is non-separable for every infinite-dimensional Banach space  $X$  is a well-known open problem.

In Section 2 we introduce the notion of a quasisubsymmetric basic sequence, give examples, and prove our main results. We say that a basic sequence  $(e_n)$  is *quasisubsymmetric* if for any two increasing sequences  $(k_n)$  and  $(\ell_n)$  of positive integers with  $k_n \leq \ell_n$  for all  $n$ , we have that  $(e_{k_n})$  dominates  $(e_{\ell_n})$ . One of our main results (Theorem 2.4) is that if a Banach space  $X$  has a seminormalized quasisubsymmetric basis then  $\ell_\infty$  embeds in  $\mathcal{L}(X)$ . Thus a reflexive Banach space  $X$  with a seminormalized quasisubsymmetric basis cannot satisfy the

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assumptions of the above problem of Banach. Moreover, in Proposition 2.3 we prove that if a Banach space  $X$  has a seminormalized basis which dominates all of its subsequences then  $\mathcal{L}(X)$  is non-separable (we do not know whether  $\ell_\infty \hookrightarrow \mathcal{L}(X)$  in this case). Finally we observe how our results provide sufficient conditions on a Banach space  $X$  which ensure that  $\ell_1$  embeds complementably into the space of nuclear operators of  $X$ .

In the rest of Section 1 we review some known results about embedding  $\ell_\infty$  into the space of operators. If  $X, Y$  are Banach spaces then denote by  $\mathcal{L}(X, Y)$  the space of all bounded operators from  $X$  to  $Y$  and  $\mathcal{K}(X, Y)$  the space of all compact operators from  $X$  to  $Y$ .

N.J. Kalton [K, proof of Theorem 6 (iii)  $\Rightarrow$  (ii)] proved that if  $X$  is an infinite-dimensional separable Banach space and  $Y$  is any Banach space then:

- (a) if  $c_0$  embeds in  $\mathcal{L}(X, Y)$  then  $\ell_\infty$  embeds in  $\mathcal{L}(X, Y)$ ,
- (b) if  $c_0$  embeds in  $Y$  then  $\ell_\infty$  embeds in  $\mathcal{L}(X, Y)$ .

Later M. Feder [F, page 201] noticed that by the result of B. Josefson [J] and A. Nissenzweig [N], the proof of Kalton for the above results works without the assumption that  $X$  is separable. Notice that the result (a) of Kalton as extended by Feder generalizes the result of C. Bessaga and A. Pelczynski [BP] stating that if  $c_0$  embeds in  $X^*$  then  $\ell_\infty$  embeds in  $X^*$ .

Recall that a sequence  $(e_n)_n$  of non-zero elements of a Banach space is called an *unconditional basic sequence* if there exists a constant  $C$  such that  $\|\sum_{n \in F} a_n e_n\| \leq C \|\sum_n a_n e_n\|$  for every finitely supported sequence of scalars  $(a_n)$  and every finite subset  $F$  of the integers, or equivalently, there exists a constant  $K$  such that  $\|\sum a_n e_n\| \leq K \|\sum b_n e_n\|$  for every finitely supported sequences of scalars  $(a_n)$  and  $(b_n)$  with  $|a_n| \leq |b_n|$  for all  $n$ . The smallest such constant  $K$  is called the *unconditionality constant* of  $(e_n)$ . Also recall that a series  $\sum_n x_n$  in a Banach space is said to converge *unconditionally* if  $\sum_n a_n x_n$  converges for any bounded sequence of scalars  $(a_n)$ . A basic sequence which is not unconditional is called a *conditional basic sequence*. A.E. Tong and D.R. Wilken [TW] proved that if  $X, Y$  are Banach spaces,  $Y$  has an unconditional basis and there is a noncompact operator in  $\mathcal{L}(X, Y)$  then  $\ell_\infty$  embeds in  $\mathcal{L}(X, Y)$ . The main focus of the paper of Tong and Wilken is the well-known open problem of whether  $\mathcal{K}(X, Y) \subsetneq \mathcal{L}(X, Y)$  implies that  $\mathcal{K}(X, Y)$  is not complemented in  $\mathcal{L}(X, Y)$ . They prove that the answer is affirmative if  $Y$  has an unconditional basis.

A.E. Tong [T] proved that if  $X$  has an unconditional basis,  $Y$  is any Banach space, and there exists a noncompact operator in  $\mathcal{L}(X, Y^*)$  then  $c_0$  embeds in  $\mathcal{L}(X, Y^*)$ . This result was generalized by Kalton [K] as follows: Let  $X$  be a Banach space with an unconditional finite-dimensional expansion of the identity (i.e. there is a sequence  $(A_n) \subseteq \mathcal{L}(X)$  of finite-rank operators such that for every  $x \in X$ ,  $x = \sum_{n=1}^\infty A_n x$  unconditionally), and let  $Y$  be any Banach space. If there is a noncompact operator in  $\mathcal{L}(X, Y)$  then  $\ell_\infty$  embeds in  $\mathcal{L}(X, Y)$ . Moreover, the same assumptions imply that  $\mathcal{K}(X, Y)$  is not complemented in  $\mathcal{L}(X, Y)$ .

G. Emmanuele [E2] proved the following result. Assume that  $X$  and  $Y$  satisfy one of the following two assumptions:

- (i)  $X$  is a  $\mathcal{L}_\infty$  space and  $Y$  is a closed subspace of a  $\mathcal{L}_1$  space, or
- (ii)  $X = C[0, 1]$  and  $Y$  is a space with cotype 2.

If there is a noncompact operator in  $\mathcal{L}(X, Y)$  then  $\ell_\infty$  embeds in  $\mathcal{L}(X, Y)$ . Moreover, under the same assumptions  $\mathcal{K}(X, Y)$  is not complemented in  $\mathcal{L}(X, Y)$ . Related results are also found in [E1].

## 2. QUASISUBSYMMETRIC SEQUENCES

In this section we introduce the notion of basic quasisubsymmetric sequences and then give examples and study their properties. The main results are Theorems 2.4 and 2.9 and Corollary 2.14. Let  $c_{00}$  denote the linear space of finitely supported sequences. If  $(x_n)$  and  $(y_n)$  are two basic sequences in a Banach space we say that  $(x_n)$  dominates  $(y_n)$  if there exists  $C > 0$  such that  $\|\sum a_n y_n\| \leq C \|\sum a_n x_n\|$  for all  $(a_n) \in c_{00}$ .

**Definition 2.1.** A basic sequence  $(e_n)_{n \in \mathbb{N}}$  in some Banach space  $X$  is said to be quasisubsymmetric if for any two increasing sequences  $(k_n)$  and  $(\ell_n)$  of positive integers with  $k_n \leq \ell_n$  for all  $n$ , we have that  $(e_{k_n})$  dominates  $(e_{\ell_n})$ , i.e. there exists a constant  $S_{(k_n),(\ell_n)} > 0$  such that

$$\left\| \sum a_n e_{\ell_n} \right\| \leq S_{(k_n),(\ell_n)} \left\| \sum a_n e_{k_n} \right\| \text{ for all } (a_n) \in c_{00}.$$

Notice that a quasisubsymmetric sequence is automatically bounded. While in this definition the constant  $S_{(k_n),(\ell_n)}$  depends on the two subsequences, the dependence can be removed if  $(e_n)$  is seminormalized (i.e. there exist positive constants  $d, D$  such that  $d \leq \|e_n\| \leq D$  for all  $n$ ), as the next result shows.

**Lemma 2.2.** Let  $(e_n)$  be a seminormalized quasisubsymmetric basic sequence. Then there exists a constant  $S > 0$  such that for any two increasing sequences  $(k_n)$  and  $(\ell_n)$  of positive integers with  $k_n \leq \ell_n$  for all  $n$ , we have that

$$(1) \quad \left\| \sum a_n e_{\ell_n} \right\| \leq S \left\| \sum a_n e_{k_n} \right\| \text{ for all } (a_n) \in c_{00}.$$

*Proof.* We will assume, to obtain a contradiction, that for all  $S > 0$  there exist  $(k_n)$  and  $(\ell_n)$  increasing sequences of positive integers with  $k_n \leq \ell_n$  for all  $n$ , and there exists  $(a_n) \in c_{00}$  for which (1) is not valid.

**Claim:** For every  $S > 0$  and every  $N \in \mathbb{N}$  there exists  $(a_n) \in c_{00}$  with  $a_n = 0$  for all  $n < N$  and there exist two increasing sequences  $(k_n)$  and  $(\ell_n)$  of positive integers with  $k_n \leq \ell_n$  for all  $n$ , such that

$$\left\| \sum a_n e_{\ell_n} \right\| > S \left\| \sum a_n e_{k_n} \right\|.$$

To prove this we fix  $S$  and  $N$  and let  $\tilde{S} = S(1+B) + (N-1)2BD/d$ , where  $B$  is the basis constant of the basic sequence  $(e_n)$  and  $d \leq \|e_n\| \leq D$  for all  $n$ . By assumption there exists  $(a_n) \in c_{00}$  and two increasing sequences  $(k_n)$  and  $(\ell_n)$  of positive integers with  $k_n \leq \ell_n$  for all  $n$ , such that

$$\left\| \sum a_n e_{\ell_n} \right\| > \tilde{S} \left\| \sum a_n e_{k_n} \right\|.$$

We can assume without loss of generality that  $\left\| \sum a_n e_{k_n} \right\| = 1$ . By the triangle inequality,

$$(2) \quad \left\| \sum_{n=N}^{\infty} a_n e_{k_n} \right\| \leq 1 + B.$$

Also,

$$(3) \quad |a_n| \leq 2B/d \text{ for all } n.$$

Furthermore,

$$\begin{aligned}
\left\| \sum_{n=N}^{\infty} a_n e_{\ell_n} \right\| &\geq \left\| \sum_{n=1}^{\infty} a_n e_{\ell_n} \right\| - \left\| \sum_{n=1}^{N-1} a_n e_{\ell_n} \right\| \\
&> \tilde{S} \left\| \sum_{n=1}^{\infty} a_n e_{k_n} \right\| - (N-1)2BD/d \text{ (by (3))} \\
&\geq S(1+B) + (N-1)2BD/d - (N-1)2BD/d \\
&\geq S \left\| \sum_{n=N}^{\infty} a_n e_{k_n} \right\| \text{ (by (2))},
\end{aligned}$$

which finishes the proof of the claim.

Let  $N_1 = 1$  and  $S_1 = 2$ . By the claim there exists an  $(a_n^{(1)}) \in c_{00}$  and two increasing sequences  $(k_n^{(1)})$  and  $(\ell_n^{(1)})$  of positive integers with  $k_n^{(1)} \leq \ell_n^{(1)}$  for all  $n$ , such that

$$\left\| \sum a_n^{(1)} e_{\ell_n^{(1)}} \right\| > 2 \left\| \sum a_n^{(1)} e_{k_n^{(1)}} \right\| = 2.$$

Let  $N_2 = \max(\ell_n^{(1)}) + 1$  where the maximum is taken over all  $n$  in the support of  $(a_n^{(1)})$  and let  $S_2 = 4$ . By the claim there exists an  $(a_n^{(2)}) \in c_{00}$  with  $a_n^{(2)} = 0$  for all  $n < N_2$  and two increasing sequences  $(k_n^{(2)})$  and  $(\ell_n^{(2)})$  of positive integers with  $k_n^{(2)} \leq \ell_n^{(2)}$  for all  $n$ , such that

$$\left\| \sum a_n^{(2)} e_{\ell_n^{(2)}} \right\| > 4 \left\| \sum a_n^{(2)} e_{k_n^{(2)}} \right\| = 4.$$

Continue in this manner with  $N_m = \max(\ell_n^{(m-1)}) + 1$  where the maximum is taken over all  $n$  in the support of  $(a_n^{(m-1)})$  and let  $S_m = 2^m$ . Notice that  $N_m \leq n \leq k_n^{(m)} \leq \ell_n^{(m)} < N_{m+1}$  for all  $m \in \mathbb{N}$  and for all  $n$  in the support of  $(a_n^{(m)})$ . Thus we can string together the sequences  $(k_n^{(m)})$  and  $(\ell_n^{(m)})$  and produce increasing sequences by setting  $a_n = a_n^{(m)}/2^m$ ,  $k_n = k_n^{(m)}$  and  $\ell_n = \ell_n^{(m)}$  for all  $m \in \mathbb{N}$  and for all  $n$  in the support of  $(a_n^{(m)})$ . Note that  $\sum a_n e_{k_n}$  converges and has norm at most equal to 2. However  $\sum a_n e_{\ell_n}$  does not converge, contradicting the fact that  $(e_n)$  is quasisubsymmetric.  $\square$

The authors do not know whether every seminormalized quasisubsymmetric sequence can be equivalently renormed so that the constant  $S$  in the statement of Lemma 2.2 can be taken to be equal to 1. If this is true then the proof of Theorem 2.4 can be slightly simplified.

Some examples of spaces with quasisubsymmetric basic sequences follow.

- (a) If any two subsequences of a basic sequence  $(e_n)$  are naturally isomorphic then  $(e_n)$  is quasisubsymmetric; (the examples (b) and (e) below satisfy this condition). In particular any subsymmetric sequence is quasisubsymmetric.
- (b) There exist quasisubsymmetric basic sequences which are conditional (refer to the Introduction for the definition of conditional basic sequences). Such an example is the summing basis, which is the completion of  $c_{00}$  equipped with the norm  $\left\| \sum a_n e_n \right\| = \sup_N \left| \sum_{n=1}^N a_n \right|$ .
- (c) The sequence of the biorthogonal functionals of the basis of Schreier's space (see e.g. [CS]) is quasisubsymmetric. We say that a finite subset  $E$  of  $\mathbb{N}$  is a *Schreier set* if the cardinality of  $E$  is less than or equal to the minimum of  $E$ . For  $a = (a_n) \in c_{00}$

we define the norm

$$\|a\| = \sup_{E \text{ is a Schreier set}} \sum_{n \in E} |a_n|.$$

Schreier's space is the completion of  $c_{00}$  equipped with this norm.

- (d) The sequence of the biorthogonal functionals of the basis of Tsirelson's space (see [FJ] and [CS]) is quasisubsymmetric. There exists a norm  $\|\cdot\|$  satisfying

$$\|x\| = \max \left\{ \|x\|_{c_0}, \frac{1}{2} \sup \sum_{j=1}^k \|E_j x\| \right\}$$

for all  $x \in c_{00}$ , where  $\|\cdot\|_{c_0}$  denotes the norm of  $c_0$  and the supremum is taken over all sequences of sets  $(E_j)_{j=1}^k$  such that  $\max E_j < \min E_{j+1}$  and  $(\min E_j)_{j=1}^k$  is a Schreier set. Also  $E_j x$  denotes the natural projection of  $x$  on  $E_j$ . Tsirelson's space is the completion of  $c_{00}$  equipped with this norm.

- (e) The James space has a boundedly complete basis when the norm is defined as

$$\|x\| = \sup \left( \sum_{k=0}^n \left( \sum_{i=p_k}^{p_{k+1}-1} x_i \right)^2 \right)^{1/2}$$

where  $x = (x_i)$  and the supremum is taken over all positive integers  $n$  and all sequences of integers such that  $0 = p_0 < p_1 < \dots < p_{n+1}$ . Since the basis is boundedly complete [FG, page 50]  $c_0 \not\hookrightarrow J$ . Also it is conditional [FG, pages 49-50]. Obviously every two subsequences of the basis are isometric. Thus our Theorem 2.4 gives an easy way to see that  $\ell_\infty \hookrightarrow \mathcal{L}(J)$ .

One of our main results of this section is Theorem 2.4 which gives sufficient conditions on  $X$  for  $\ell_\infty \hookrightarrow \mathcal{L}(X)$ . If, however, we are only interested in  $\mathcal{L}(X)$  being non-separable, then the assumptions of Theorem 2.4 can be weakened as the next result shows.

**Proposition 2.3.** *If a Banach space  $X$  has a seminormalized basis which dominates its subsequences then  $\mathcal{L}(X)$  is non-separable.*

*Proof.* Let  $(e_n)$  be a seminormalized basis for  $X$  such that it dominates its subsequences and let  $C$  be its basis constant (see the Introduction for the definition of the basis constant). If  $(k_n)$  is an increasing sequence in  $\mathbb{N}$  then define  $T_{(k_n)} : X \rightarrow X$  by  $T_{(k_n)}(x) = \sum_{n=1}^{\infty} e_n^*(x) e_{k_n}$ . By our assumption each  $T_{(k_n)}$  is bounded, i.e.  $T_{(k_n)} \in \mathcal{L}(X)$ . Let  $(k_n), (m_n)$  be two different increasing sequences in  $\mathbb{N}$ . Then there exists some  $j \in \mathbb{N}$  such that  $k_j \neq m_j$ . Thus

$$\begin{aligned} \|T_{(k_n)} - T_{(m_n)}\| &\geq \left\| \sum_{n=1}^{\infty} \left( e_n^* \left( \frac{e_j}{\|e_j\|} \right) e_{k_n} - e_n^* \left( \frac{e_j}{\|e_j\|} \right) e_{m_n} \right) \right\| \\ &= \frac{1}{\|e_j\|} \|e_{k_j} - e_{m_j}\| \geq \frac{\|e_{k_j} - e_{m_j}\|}{\sup_n \|e_n\|} \geq \frac{\inf_n \|e_n\|}{C \sup_n \|e_n\|} > 0. \end{aligned}$$

□

We now present our first main result.

**Theorem 2.4.** *If a Banach space  $X$  has a seminormalized quasisubsymmetric basis then  $\ell_\infty \hookrightarrow \mathcal{L}(X)$ .*

In the proof of Theorem 2.4, we will use the following remark due to Tong and Wilken [TW, Proposition 4] which is mentioned in the Introduction.

**Remark 2.5.** *Let  $(w_n)$  be an unconditional basic sequence in a Banach space  $X$ , and assume that there exists a noncompact operator in  $\mathcal{L}(X, [(w_n)])$ . Then  $\ell_\infty \hookrightarrow \mathcal{L}(X)$ .*

*Proof of Theorem 2.4.* Let  $(e_n)_{n \in \mathbb{N}}$  be a seminormalized quasisubsymmetric basis for  $X$ . Since  $(e_n)$  is bounded, by the  $\ell_1$  theorem of H.P. Rosenthal [R] there exists a subsequence  $(e_{k_n})_{n \in \mathbb{N}}$  which is either equivalent to the unit vector basis of  $\ell_1$  or it is weakly Cauchy.

In the first case, for some  $K$  and for all  $(a_n) \in c_{00}$  we have

$$\frac{K}{S} \sum |a_n| \leq \frac{1}{S} \left\| \sum_{n=1}^{\infty} a_n e_{k_n} \right\| \leq \left\| \sum_{n=1}^{\infty} a_n e_n \right\| \leq \sup_n \|e_n\| \sum_{n=1}^{\infty} |a_n|$$

where  $S$  is the constant provided by Lemma 2.2. Thus  $(e_n)$  is equivalent to the unit vector basis of  $\ell_1$ , hence it is an unconditional basis for  $X$  and therefore  $(e_n^* \otimes e_n)_{n=1}^{\infty}$  is equivalent to the unit vector basis of  $c_0$  (where for  $x \in X$  and  $x^* \in X^*$ ,  $x^* \otimes x$  denotes the operator on  $X$  defined by  $(x^* \otimes x)(y) = x^*(y)x$  for every  $y \in X$ ).

In the second case we assume that  $(e_{k_n})_{n=1}^{\infty}$  is weakly Cauchy. Define the difference sequence  $(z_n)$  of  $(e_{k_n})$  by  $z_n = e_{k_{2n}} - e_{k_{2n-1}}$ . Then  $(z_n)_{n \in \mathbb{N}}$  is weakly null. Since  $(e_n)$  is a seminormalized basic sequence, we have that  $(z_n)$  is also a seminormalized basic sequence, say  $0 < d \leq \|z_n\| \leq D < \infty$  for all  $n$ . We claim that there exists a subsequence of  $(z_n)_{n \in \mathbb{N}}$  which is unconditional. In order to prove this, we use an argument of A. Brunel and L. Sucheston [BS]. First by Mazur's theorem [W, Corollary II.A.5] there exist intervals of the natural numbers  $I_1 < I_2 < \dots$  (where  $I_k < I_j$  means  $\max(I_k) < \min(I_j)$ ) and there exist  $c_i \geq 0$  with  $\sum_{i \in I_j} c_i = 1$  for all  $j$ , such that  $\|\sum_{i \in I_j} c_i z_i\| \rightarrow 0$  as  $j \rightarrow \infty$ .

By taking a subsequence and renaming we can assume without loss of generality that

$$(4) \quad \left\| \sum_{i \in I_n} c_i z_i \right\| < \frac{1}{2^n}.$$

Let  $(a_n) \in c_{00}$ . Let  $F$  be a finite subset of positive integers. For  $n \in \mathbb{N}$  let  $m_n := \min I_n$ . We will show

$$(5) \quad \left\| \sum_{n \in \mathbb{N} \setminus F} a_n z_{m_n} \right\| \leq (S + \frac{2C}{d}) \left\| \sum_{n=1}^{\infty} a_n z_{m_n} \right\| \text{ for all } (a_n) \in c_{00}$$

where  $C$  is the basis constant of  $(z_n)$  and  $S$  is the constant provided by Lemma 2.2 for the basic sequence  $(e_n)$ . First, note since  $(e_n)$  is quasisubsymmetric, that for all sequences  $(\ell_n)$  of positive integers with  $\ell_n \in I_n$  (for all  $n$ ), and for all  $(a_n) \in c_{00}$ ,

$$\left\| \sum_{n=1}^{\infty} a_n z_{\ell_n} \right\| \leq S \left\| \sum_{n=1}^{\infty} a_n z_{m_n} \right\|.$$

Let  $(a_n) \in c_{00}$ . Assume that  $F = \{t_1, t_2, \dots, t_s\}$ . For any  $(i_1, i_2, \dots, i_s)$  with  $i_n \in I_{t_n}$  for  $n = 1, \dots, s$ , define

$$V(i_1, i_2, \dots, i_s) = \sum_{n \in \mathbb{N} \setminus F} a_n z_{m_n} + \sum_{n=1}^s a_{t_n} z_{i_n}.$$

Thus if we sum over all  $(i_1, i_2, \dots, i_s)$  with  $i_n \in I_{t_n}$  for  $n = 1, \dots, s$ , we obtain

$$(6) \quad \sum c_{i_1} \cdots c_{i_s} V(i_1, \dots, i_s) = \sum_{n \in \mathbb{N} \setminus F} a_n z_{m_n} + \sum_{n=1}^s a_{t_n} \sum_{i \in I_{t_n}} c_i z_i.$$

Apply Lemma 2.2 for  $k_n = m_n$  if  $n \in \mathbb{N}$ ,  $\ell_n = m_n$  if  $n \in \mathbb{N} \setminus F$  and  $\ell_n = i_n$  if  $n \in F$ , to obtain

$$\|V(i_1, \dots, i_s)\| \leq S \left\| \sum_{n=1}^{\infty} a_n z_{m_n} \right\|.$$

Thus

$$(7) \quad \begin{aligned} \left\| \sum_{i_n \in I_{t_n} \text{ for } n=1, \dots, s} c_{i_1} \cdots c_{i_s} V(i_1, \dots, i_s) \right\| &\leq \sum_{i_n \in I_{t_n} \text{ for } n=1, \dots, s} c_{i_1} \cdots c_{i_s} \|V(i_1, \dots, i_s)\| \\ &\leq \sum_{i_n \in I_{t_n} \text{ for } n=1, \dots, s} c_{i_1} \cdots c_{i_s} S \left\| \sum_{n=1}^{\infty} a_n z_{m_n} \right\| \\ &= S \left\| \sum_{n=1}^{\infty} a_n z_{m_n} \right\|. \end{aligned}$$

Also note

$$(8) \quad |a_n|d \leq \|a_n z_{m_n}\| = \left\| \sum_{i=1}^n a_i z_{m_i} - \sum_{i=1}^{n-1} a_i z_{m_i} \right\| \leq 2C \left\| \sum_{i=1}^{\infty} a_i z_{m_i} \right\|.$$

Hence by (6),

$$(9) \quad \begin{aligned} \left\| \sum_{i_n \in I_{t_n} \text{ for } n=1, \dots, s} c_{i_1} \cdots c_{i_s} V(i_1, \dots, i_s) \right\| &\geq \left\| \sum_{n \in \mathbb{N} \setminus F} a_n z_{m_n} \right\| - \sum_{n=1}^s |a_{t_n}| \left\| \sum_{i \in I_{t_n}} c_i z_i \right\| \\ &\geq \left\| \sum_{n \in \mathbb{N} \setminus F} a_n z_{m_n} \right\| - \frac{2C}{d} \left\| \sum_{i=1}^{\infty} a_i z_{m_i} \right\| \sum_{n=1}^s \frac{1}{2^{t_n}} \\ &\quad (\text{by (4) and (8)}) \\ &\geq \left\| \sum_{n \in \mathbb{N} \setminus F} a_n z_{m_n} \right\| - \frac{2C}{d} \left\| \sum_{n=1}^{\infty} a_n z_{m_n} \right\|. \end{aligned}$$

So by (7) and (9) we obtain (5) which proves that  $(z_{m_n})$  is unconditional.

Let  $P \in \mathcal{L}(X, [(z_{m_n})])$  be defined by

$$P \left( \sum_{n=1}^{\infty} a_n e_n \right) = \sum_{n=1}^{\infty} a_n z_{m_n}.$$

Notice that  $P$  is bounded since

$$\begin{aligned} \left\| P \left( \sum_{n=1}^{\infty} a_n e_n \right) \right\| &\leq \|a_1 e_{k_{2m_1}} + a_2 e_{k_{2m_2}} + \cdots\| + \|a_1 e_{k_{2m_1-1}} + a_2 e_{k_{2m_2-1}} + \cdots\| \\ &\leq 2S \left\| \sum_{n=1}^{\infty} a_n e_n \right\|. \end{aligned}$$

Since  $P$  is a noncompact operator and  $(z_{m_n})$  is unconditional, by Remark 2.5, we have  $\ell_{\infty} \hookrightarrow \mathcal{L}(X)$ .  $\square$

By observing that for a Banach space  $X$  the map from  $\mathcal{L}(X)$  to  $\mathcal{L}(X^*)$  given by  $T \mapsto T^*$ , where  $T^*$  is the adjoint operator, is an isometric embedding, Theorem 2.4 gives the following corollary.

**Corollary 2.6.** *If a Banach space  $X$  has a seminormalized quasisubsymmetric basis then  $\ell_{\infty} \hookrightarrow \mathcal{L}(X^*)$ .*

The proof of Theorem 2.4 gives the following two corollaries:

**Corollary 2.7.** *If  $(e_n)_{n \in \mathbb{N}}$  is a seminormalized quasisubsymmetric basic sequence then either  $(e_n)_{n \in \mathbb{N}}$  is equivalent to the unit vector basis of  $\ell_1$  or there exists a subsequence  $(e_{k_n})$  such that  $(e_{k_{2n}} - e_{k_{2n-1}})_{n \in \mathbb{N}}$  is unconditional.*

**Corollary 2.8.** *If  $(e_n)_{n \in \mathbb{N}}$  is a seminormalized quasisubsymmetric weakly null basic sequence then there exists a subsequence  $(e_{k_n})$  of  $(e_n)$  which is unconditional.*

**Theorem 2.9.** *If a Banach space  $X$  has a seminormalized basis  $(e_n)$  and the sequence  $(e_n^*)$  of the biorthogonal functionals is quasisubsymmetric then  $\ell_{\infty} \hookrightarrow \mathcal{L}(X)$ .*

*Proof.* By results of Kalton [K] mentioned in the Introduction, it suffices to show that  $c_0 \hookrightarrow \mathcal{L}(X)$ . By Corollary 2.7 we separate two cases:

If  $(e_n^*)_{n \in \mathbb{N}}$  is equivalent to the unit vector basis of  $\ell_1$ , we know that  $X = [(e_n)_{n \in \mathbb{N}}]$  is isomorphic to  $c_0$ . Therefore  $c_0 \hookrightarrow \mathcal{L}(X)$ .

If  $(e_{k_{2n}}^* - e_{k_{2n-1}}^*)_{n \in \mathbb{N}}$  is unconditional, let  $K$  be its unconditionality constant and proceed by showing that  $((e_{k_{2n}}^* - e_{k_{2n-1}}^*) \otimes e_n)_{n \in \mathbb{N}}$  is equivalent to the unit vector basis of  $c_0$ : For  $(a_n)_{n \in \mathbb{N}} \in c_{00}$  choose  $N \in \mathbb{N}$  such that  $a_n = 0$  for  $n > N$  and define the operator

$$T := \sum_{n=1}^N a_n (e_{k_{2n}}^* - e_{k_{2n-1}}^*) \otimes e_n : X \rightarrow [(e_n)_{n=1}^N].$$

Then

$$T^* = \sum_{n=1}^N a_n e_n \otimes (e_{k_{2n}}^* - e_{k_{2n-1}}^*) : [(e_n)_{n=1}^N]^* \rightarrow X^*,$$

where  $e_n$  is considered as an element of  $X^{**}$ . Hence

$$(10) \quad \left\| \sum_{n=1}^N a_n (e_{k_{2n}}^* - e_{k_{2n-1}}^*) \otimes e_n \right\| = \|T\| = \|T^*\|.$$

Let  $x^* \in [(e_n)_{n=1}^N]^*$  with  $\|x^*\| = 1$  and  $\|T^*\| \leq 2\|T^*x^*\|$ . Let  $y^* \in [(e_n^*)_{n=1}^N]$  be defined by  $y^* = \sum_{n=1}^N x^*(e_n) e_n^*$ . Then  $y^*(e_n) = x^*(e_n)$  for  $n = 1, \dots, N$  and  $\|y^*\| \leq C$  where  $C$  denotes



the basis constant of  $(e_n)$ . Thus

$$(11) \quad \|T^*\| \leq 2\|T^*x^*\| = 2\left\|\sum_{n=1}^N a_n e_n(y^*)(e_{k_{2n}}^* - e_{k_{2n-1}}^*)\right\|.$$

Now use the facts that  $(e_{k_{2n}}^* - e_{k_{2n-1}}^*)_{n \in \mathbb{N}}$  is unconditional with unconditionality constant  $K$  (see the Introduction for the definition of the unconditionality constant) and  $(e_n^*)$  is quasisubsymmetric to obtain:

$$(12) \quad \begin{aligned} 2\left\|\sum_{n=1}^N a_n e_n(y^*)(e_{k_{2n}}^* - e_{k_{2n-1}}^*)\right\| &\leq 2K \max_{1 \leq n \leq N} |a_n| \left\|\sum_{n=1}^N e_n(y^*)(e_{k_{2n}}^* - e_{k_{2n-1}}^*)\right\| \\ &\leq 2K\|(a_n)\|_{c_0} \left( \left\|\sum_{n=1}^N e_n(y^*)e_{k_{2n}}^*\right\| + \left\|\sum_{n=1}^N e_n(y^*)e_{k_{2n-1}}^*\right\| \right) \\ &\leq 2K\|(a_n)\|_{c_0} \left( S_{(n),(k_{2n})} \left\|\sum_{n=1}^N e_n(y^*)e_n^*\right\| + S_{(n),(k_{2n-1})} \left\|\sum_{n=1}^N e_n(y^*)e_n^*\right\| \right) \\ &= 2K(S_{(n),(k_{2n})} + S_{(n),(k_{2n-1})})\|y^*\| \|(a_n)\|_{c_0} \quad (\text{since } y^* = \sum_{n=1}^N e_n(y^*)e_n^*) \\ &\leq 2KC(S_{(n),(k_{2n})} + S_{(n),(k_{2n-1})})\|(a_n)\|_{c_0}. \end{aligned}$$

Equations (10), (11) and (12) show that  $((e_{k_{2n}}^* - e_{k_{2n-1}}^*) \otimes e_n)_{n \in \mathbb{N}}$  is dominated by the  $c_0$  basis. On the other hand, for  $1 \leq n \leq N$ ,  $Te_{k_{2n}} = a_n e_n$ . Thus

$$\left\|\sum_{n=1}^N a_n (e_{k_{2n}}^* - e_{k_{2n-1}}^*) \otimes e_n\right\| = \|T\| \geq \frac{\inf_i \|e_i\|}{\sup_i \|e_i\|} \|(a_n)\|_{c_0}$$

which implies that  $((e_{k_{2n}}^* - e_{k_{2n-1}}^*) \otimes e_n)_{n \in \mathbb{N}}$  dominates the  $c_0$  basis. Hence  $c_0 \hookrightarrow \mathcal{L}(X)$ .  $\square$

**Corollary 2.10.** *If a Banach space  $X$  has a seminormalized quasisubsymmetric basis  $(e_n)$  then  $\ell_\infty \hookrightarrow \mathcal{L}([(e_n^*)])$ , where  $(e_n^*)$  is the sequence of the biorthogonal functionals to  $(e_n)$ .*

*Proof.* Consider  $(e_n)$  in  $[(e_n^*)]^*$ : it is seminormalized, quasisubsymmetric and biorthogonal to  $(e_n^*)$ . Therefore by Theorem 2.9 we have  $\ell_\infty \hookrightarrow \mathcal{L}([(e_n^*)])$ .  $\square$

Finally we make some remarks related to the question of whether  $\ell_1$  embeds into the space of nuclear operators of a Banach space  $X$ . If  $X$  is a Banach space let  $(\mathcal{N}(X), \nu(\cdot))$  denote the space of nuclear operators on  $X$ , namely

$$\mathcal{N}(X) = \{T \in \mathcal{L}(X) : T = \sum_{n=1}^{\infty} x_n^* \otimes y_n \text{ where } x_n^* \in X^*, y_n \in X \text{ and } \sum_{n=1}^{\infty} \|x_n^*\| \|y_n\| < \infty\}$$

$$\nu(T) = \inf \left\{ \sum_{n=1}^{\infty} \|x_n^*\| \|y_n\| : \sum_{n=1}^{\infty} x_n^* \otimes y_n \text{ is a representation of } T \right\}.$$

The following remark is well-known:

**Remark 2.11.** *If  $X$  is a Banach space with the AP then  $\mathcal{N}(X)^*$  is isometric to  $\mathcal{L}(X^*)$ . Moreover, if  $T \in \mathcal{L}(X^*)$  then the action of  $T$  as a functional on  $\mathcal{N}(X)$  gives the trace of  $TS^*$ ,  $tr(TS^*)$ , when it is applied to  $S \in \mathcal{N}(X)$ .*

Indeed, it is shown in [Ry, Proposition 4.6 (i)  $\Leftrightarrow$  (ii), cf. Corollary 4.8 (a)] that if the Banach space  $X$  has the AP then  $\mathcal{N}(X)$  is naturally isometric to the projective tensor product  $X^* \hat{\otimes} X$ . Also in [Ry, page 24] it is shown that for any Banach space  $Y$ ,  $(Y^* \hat{\otimes} Y)^*$  is isometric to  $\mathcal{L}(Y^*)$ , where for  $T \in \mathcal{L}(Y^*)$ , the action of  $T$  as a functional on  $\sum_{i=1}^n y_i^* \otimes y_i \in Y^* \hat{\otimes} Y$  gives  $\sum_{i=1}^n (Ty_i^*)(y_i)$ .

By Remark 2.11 we have that for a Banach space  $X$  with the AP,  $\ell_1$  embeds complementably in  $\mathcal{N}(X)$  if and only if  $\ell_\infty$  embeds in  $\mathcal{L}(X^*)$ .

It is easy to see (and also follows from the result of Tong [T, Theorem 1.5] mentioned in the Introduction) that for a dual Banach space  $X^*$  with an unconditional basis, there exists a noncompact operator in  $\mathcal{L}(X^*)$  if and only if  $\ell_1$  embeds in  $X^* \hat{\otimes} X$  complementably. The next remarks give sufficient conditions on a Banach space  $X$  which ensure that  $\ell_1$  embeds complementably into  $\mathcal{N}(X)$ .

**Remark 2.12.** *Let  $X$  be a Banach space which has the AP and assume that  $\ell_\infty \hookrightarrow \mathcal{L}(X)$ . Then  $\ell_1 \hookrightarrow \mathcal{N}(X)$  complementably.*

*Proof.* Since the map from  $\mathcal{L}(X)$  to  $\mathcal{L}(X^*)$  given by  $T \mapsto T^*$  is an isometric embedding, we have that  $\ell_\infty$  embeds in  $\mathcal{L}(X^*)$ . Hence by [BP] and Remark 2.11 we obtain that  $\ell_1$  embeds complementably in  $\mathcal{N}(X)$ .  $\square$

**Remark 2.13.** *Let  $X$  be a Banach space which has the AP and contains an unconditional basic sequence  $(e_n)$ . Suppose that there exists a noncompact operator in  $\mathcal{L}(X, [(e_n)])$ . Then  $\ell_1 \hookrightarrow \mathcal{N}(X)$  complementably.*

*Proof.* By Remark 2.5 we have that  $\ell_\infty$  embeds in  $\mathcal{L}(X)$ . Now Remark 2.12 finishes the proof.  $\square$

**Corollary 2.14.** *If a Banach space  $X$  has a seminormalized quasisubsymmetric basis then  $\ell_1 \hookrightarrow \mathcal{N}(X)$  complementably.*

*Proof.* Combine Theorem 2.4 and Remark 2.12.  $\square$

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