Math 3160 - Test 2.5 Review

Anything previous material can be on the exam, plus the following problems.

1 Change of Basis Matrix

1. Let $B = \{(1,0),(0,1)\}$, $B_1 = \{(-1,1),(2,3)\}$ and $B_2 = \{(1,-1),(1,1)\}$.
   (a) Find the change of basis matrices for $P_{B_1 \to B_2}$ and $P_{B_2 \to B_1}$.
   (b) Find the coordinates of the point $(4,6)$ (given in the standard basis) relative to the bases B_1 and B_2.
   (c) Find the change of basis matrices for $P_{B \to B_2}$ and $P_{B_2 \to B}$.
   (d) Find the coordinates of the point $(2,-4)$ (given in the standard basis) relative to the bases B and B_2. Graph this point the two separate coordinate axes B and B_2.

2 Eigenvalues, eigenvectors and Diagonalization

2. Find the eigenvalues and eigenvetors for the following matrices

$$A = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \text{ and } D = \begin{bmatrix} 4 & 0 & 0 \\ 3 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix}.$$

3. Find the eigenvalues and eigenvetors for the following matrices (may have complex numbers in answer).

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 3 & 3 & 0 \\ 0 & 5 & 0 \\ 1 & 0 & 5 \end{bmatrix}.$$

4. Can the following matricses be diagonalized? State why? If it can be diagonalized, do it. That is, if it is diagonalizable compute D and P.

$$A = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \text{ and } D = \begin{bmatrix} 4 & 0 & 0 \\ 3 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix}.$$

3 Inner Product Spaces

5. Let $f_1(x) = x^2$, $f_2(x) = 1 + x$, $f_3(x) = 3 - 4x$. Let

$$\langle f, g\rangle = \int_0^1 fg\, dx.$$

Compute

(a) $\langle f_1, f_2\rangle$

(b) $\langle f_1, f_1\rangle$

(c) $\|f_1\|$

(d) Find the angle between f_1 and f_2.

(e) Find the angle between f_1 and f_3.

6. Let $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = x^2$. Let

$$\langle f, g\rangle = \int_{-1}^1 fg\, dx.$$

Compute

(a) $\langle f_1, f_2\rangle$

(b) $\langle f_1, f_1\rangle$

(c) $\|f_1\|$

(d) Find the angle between f_1 and f_2.

(e) Find the angle between f_1 and f_3.

7. Let $f_1(x) = 1$, and $f_2(x) = x^2$. Let

$$\langle f, g\rangle = \int_0^1 fg\, dx.$$

Compute

(a) $\mathrm{Proj}_{f_1} f_2$

(b) $\mathbf{a} =$ $\mathrm{Proj}_{f_1} f_2$. Compute $\mathbf{b} = f_2 - \mathbf{a}$. What did we call $\mathbf{a}$ and $\mathbf{b}$ earlier in the semester?

(c) Compute the angle between $\mathbf{a}$ and $\mathbf{b}$.

4 Gram-Schmidt Orthogonalizations

8. Let $\mathbf{v_1} = (1, 0, -1)$, $\mathbf{v_2} = (0, 0, -1)$, and $\mathbf{v_3} = (0, 1, 1)$ in the usual Euclidean $\mathbb{R}^3$. Use GS to orthonormalize this set of vectors.

9. Let $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = x^2$. Let
$$\langle f, g \rangle = \int_0^1 fg\, dx.$$
Use GS to orthonormalize this set of vectors.

5 Markov Chains

10. For the following defined matrices which are stochastic. $A = \begin{bmatrix} .1 & .9 \\ .1 & .9 \end{bmatrix}$,
$B = \begin{bmatrix} .9 & 0 \\ .1 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 0.25 & -1.0 \\ 0.75 & 2.0 \end{bmatrix}$ and $D = \begin{bmatrix} .1 & .2 & .3 & .4 \\ .0 & .4 & .1 & .5 \\ .0 & .4 & .6 & 0 \\ .9 & 0 & 0 & .1 \end{bmatrix}$

11. For the following defined markov processes which are regular.
$A = \begin{bmatrix} .1 & .9 \\ .1 & .9 \end{bmatrix}$, $B = \begin{bmatrix} .9 & 0 \\ .1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 0.25 & 1.0 \\ 0.75 & 0.0 \end{bmatrix}$

12. Find the steady states for the following processes.

    (a) For the transition matrix $A = \begin{bmatrix} .1 & .9 \\ .9 & .1 \end{bmatrix}$. You may be able to guess the answer for this one (but do the work anyway).

    (b) A bike rental company has three locations. A renter can pick up a bike at any location and drop that bike off at any location.

    - A bike picked up at location A has 20% probability of being dropped off at location B and has a 10% probability of being dropped off at location C.
    - A bike picked up at location B has 30% probability of being dropped off at location A and has a 0% probability of being dropped off at location C.
    - A bike picked up at location C has 10% probability of being dropped off at location B and has a 40% probability of being dropped off at location A.

6 Inverse Matrices mod 26

13. Reduce the following mod 26.

    (a) $33 \mod 26$

    (b) $-33 \mod 26$

    (c) $100 \mod 26$

    (d) $1000 \mod 26$

14. Solve for x mod 26.

    (a) $x + 33 \equiv 7 \mod 26$

    (b) $3x \equiv 1 \mod 26$

    (c) $2x \equiv 1 \mod 26$

15. Find the inverse of the following matrices mod 26. If not possible, state why.

$$A = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 9 & 2 \\ 3 & 1 \end{bmatrix}, C = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}, \text{ and } D = \begin{bmatrix} 4 & -1 \\ 5 & 2 \end{bmatrix}.$$