

(1) (a) From Class.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{c} -5 \\ 1/2 \\ -1/2 \\ 0 \end{array} \begin{array}{c} 4 \\ 2 \\ 3 \\ 0 \end{array}$$

$x_4 = s \quad x_5 = t$

$$S = \left\{ \begin{bmatrix} 4 - 6s + 5t \\ 2 - 3s + 1/2 t \\ 3 - 3s + 1/2 t \\ s \\ t \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

(b) $\left[\begin{array}{ccc|c} 2 & -2 & 4 & 2 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 \end{array} \right] \rightarrow \dots \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 0 \end{array} \right], S = \left\{ \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix} \right\}$

(c) $\left[\begin{array}{ccc|c} 2 & -2 & 4 & 2 \\ -1 & -1 & 3 & 2 \\ 1 & -3 & 7 & 2 \end{array} \right] \rightarrow \dots \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1/2 & -1/2 \\ 0 & 1 & -5/2 & -3/2 \\ 0 & 0 & 0 & -2 \end{array} \right]$

$0 = -2 \Rightarrow$ No Solution

$S = \{ \}$

(2) (a) $\begin{cases} 2x_1 - 2x_2 + 4x_3 = 2 \\ -x_2 + 3x_3 = 2 \\ -3x_2 + 7x_3 = 2 \end{cases}$

SOLN

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 0 & -1 & 3 \\ 0 & -3 & 7 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \quad \vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

FIND A^{-1}

$$\left[\begin{array}{ccc|ccc} 2 & -2 & 4 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 0 & -3 & 7 & 0 & 0 & 1 \end{array} \right]$$

$$\frac{1}{2}R_1 \rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1/2 & 0 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 0 & -3 & 7 & 0 & 0 & 1 \end{array} \right]$$

$$\begin{array}{l} R_1 \\ -R_2 \\ R_3 \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1/2 & 0 & 0 \\ 0 & 1 & -3 & 0 & -1 & 0 \\ 0 & -3 & 7 & 0 & 0 & 1 \end{array} \right]$$

$$\begin{array}{l} R_1 + R_2 \\ R_3 + 3R_2 \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1/2 & -1 & 0 \\ 0 & 1 & -3 & 0 & -1 & 0 \\ 0 & 0 & -2 & 0 & -3 & 1 \end{array} \right]$$

$$\downarrow$$

$$-\frac{1}{2}R_3 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1/2 & -1 & 0 \\ 0 & 1 & -3 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 3/2 & -1/2 \end{array} \right]$$

$$\begin{array}{l} R_1 + R_3 \\ R_2 + 3R_3 \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 & 0 & 7/2 & -3/2 \\ 0 & 0 & 1 & 0 & 3/2 & -1/2 \end{array} \right] \Rightarrow A^{-1} = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ 0 & 7/2 & -3/2 \\ 0 & 3/2 & -1/2 \end{bmatrix}$$

2a) (CONTINUED)

$$S_0 \quad \vec{X} = A^{-1} \vec{b} = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ 0 & 7/2 & -3/2 \\ 0 & 3/2 & -1/2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2(2) + 1/2(2) - 1/2(2) \\ 7/2(2) - 3/2(2) \\ 3/2(2) - 1/2(2) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, S = \left\{ \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \right\}$$

$$(2b) \quad \begin{cases} 2x - 2y = 2 \\ x - 3y = 2 \end{cases}$$

$$A = \begin{bmatrix} 2 & -2 \\ 1 & -3 \end{bmatrix} \quad \vec{X} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\vec{X} = A^{-1} \vec{b} = \begin{bmatrix} 3/4 & -2/4 \\ 1/4 & -2/4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} \quad S = \left\{ \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} \right\}$$

$$A^{-1} = \frac{1}{(2)(-3) - 1(-2)} \begin{bmatrix} -3 & 2 \\ -1 & 2 \end{bmatrix} = \frac{1}{-4} \begin{bmatrix} -3 & 2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3/4 & -2/4 \\ 1/4 & -2/4 \end{bmatrix}$$

$$\begin{aligned} (3) \quad |A| &= \begin{vmatrix} 2 & -2 & 0 \\ -1 & 4 & 1 \\ 0 & 4 & 1 \end{vmatrix} = + (0) \begin{vmatrix} -2 & 0 \\ 4 & 1 \end{vmatrix} - (4) \begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix} + (1) \begin{vmatrix} 2 & -2 \\ -1 & 4 \end{vmatrix} \\ &= 0 - 4[2 - 0] + 1(8 - 2) = -8 + 6 = \boxed{-2} \end{aligned}$$

$$|B| = 0$$

$$|C| = 96$$

$$|D| = 8$$

$$(b) D^3 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(c) C = \begin{bmatrix} 2 & -2 & 0 & -2 & 0 \\ 0 & -2 & 0 & -2 & 0 \\ 0 & 0 & 3 & -2 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix},$$

$$C^T C = \begin{bmatrix} 4 & -4 & 0 & -4 & 0 \\ -4 & 8 & 0 & 8 & 0 \\ 0 & 0 & 9 & -6 & 0 \\ -4 & 8 & -6 & 16 & 0 \\ 0 & 0 & 0 & 0 & 16 \end{bmatrix}$$

NOTE $C^T C$ is symmetric

$$C^T C = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ -2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ -2 & -2 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2(2) + 0 + 0 + 0 + 0 \\ (-2)(-2) + (-2)(-2) + (-2)(-2) + (-2)(-2) + 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$(4) \boxed{Ax = Bx - A}$$

$$Ax - Bx = -A$$

$$(A - B)x = -A$$

$$(A - B)^{-1} (A - B)x = -(A - B)^{-1} A$$

$$\boxed{x = -(A - B)^{-1} A}$$

(b)

$$Ax = 2x - A$$

$$Ax - 2x = -A$$

$$Ax - 2Ix = -A$$

$$(A - 2I)x = -A$$

$$(A - 2I)^{-1} (A - 2I)x = (A - 2I)^{-1} (-A)$$

$$x = -(A - 2I)^{-1} A$$

$$(c) A\vec{u} = 2\vec{u} + B$$

$$A\vec{u} - 2\vec{u} = B$$

$$A\vec{u} - 2I\vec{u} = B$$

$$(A - 2I)\vec{u} = B$$

$$(A - 2I)^{-1} (A - 2I)\vec{u} = (A - 2I)^{-1} B$$

$$\boxed{\vec{u} = (A - 2I)^{-1} B}$$

$$(5) \begin{cases} 2x_1 - 2x_2 + 4x_3 = 2 \\ -x_2 + 3x_3 = 2 \\ -3x_2 + 7x_3 = 2 \end{cases}$$

$$|A| = \begin{vmatrix} 2 & -2 & 4 \\ 0 & -1 & 3 \\ 0 & -3 & 7 \end{vmatrix} = 2 \begin{vmatrix} -1 & 3 \\ -3 & 7 \end{vmatrix} - 0 \begin{vmatrix} ? & ? \\ ? & ? \end{vmatrix} + 0 \begin{vmatrix} ? & ? \\ ? & ? \end{vmatrix} = 2(-7 + 9) = 4$$

$$|A_1| = \begin{vmatrix} 2 & -2 & 4 \\ 2 & -1 & 3 \\ 2 & -3 & 7 \end{vmatrix} = 2(2) - 2(14 + 12) + 2(-6 + 4) = 4$$

$$x_1 = \frac{|A_1|}{|A|} = \frac{4}{4} = 1$$

$$|A_2| = \begin{vmatrix} 2 & 2 & 4 \\ 0 & 2 & 3 \\ 0 & 2 & 7 \end{vmatrix} = 16$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{16}{4} = 4$$

$$|A_3| = \begin{vmatrix} 2 & -2 & 2 \\ 0 & -1 & 2 \\ 0 & -3 & 2 \end{vmatrix} = 8$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{8}{4} = 2$$

$$(b) \begin{cases} 2x - 2y = 2 \\ -x - 3y = 2 \end{cases}$$

$$|A| = \begin{vmatrix} 2 & -2 \\ -1 & -3 \end{vmatrix} = -6 + 2 = -8$$

$$|A_1| = \begin{vmatrix} 2 & -2 \\ 2 & -3 \end{vmatrix} = -6 + 4 = -2$$

$$x_1 = \frac{|A_1|}{|A|} = \frac{-2}{-8} = +1/4$$

$$S = \left\{ \begin{bmatrix} 1/4 \\ -3/4 \end{bmatrix} \right\}$$

$$|A_2| = \begin{vmatrix} 2 & 2 \\ -1 & 2 \end{vmatrix} = 4 + 2 = 6$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{6}{-8} = -3/4$$

(2)

$$(6.) \begin{cases} x_1 - 2x_2 = 0 \\ x_1 + x_2 = 5 \end{cases}$$

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\vec{x} = A^{-1} \vec{b} = \begin{bmatrix} 1/3 & 2/3 \\ -1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 10/3 \\ 5/3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{1+2} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 \\ -1/3 & 1/3 \end{bmatrix}$$

$$S = \left\{ \begin{bmatrix} 10/3 \\ 5/3 \end{bmatrix} \right\}$$

$$(7) \begin{cases} x_1 - x_2 = 0 \\ x_2 + x_3 = 5 \\ 2x_2 = 4 \end{cases}$$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Find A^{-1}

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 \end{array} \right] \rightarrow \dots \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 1 & -1/2 \end{array} \right]. \text{ So } A^{-1} = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 0 & 1/2 \\ 0 & 1 & -1/2 \end{bmatrix}$$

$$\text{Thus } \vec{x} = A^{-1} \vec{b} = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 0 & 1/2 \\ 0 & 1 & -1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 + 0 + 1/2(4) \\ 0 + 0 + 1/2(4) \\ 0 + 5 - 1/2(4) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

$$S = \left\{ \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \right\}$$