Math 3160 - Gram-Schmidt, Orthogonality, Least Squares, and QR Decomposition

1. Find the Gram-Schmidt orthogonalization of the following vestors using the usual inner product.

$$\mathbf{v_1} = (0,0,3), \mathbf{v_2} = (1,0,-2), \mathbf{v_3} = (1,2,3)$$

2. Find the Gram-Schmidt orthogonalization of the follwing vectors

$$f_1(x) = 1, f_2(x) = 1 - x, f_3(x) = 1 + x^2$$

with the following inner product.

$$\langle f(x), g(x) \rangle = \int_0^1 f(x)g(x)\,dx$$

3. Find the Gram-Schmidt orthogonalization of the column vecors of A using the usual inner product. And find the QR decomposition of A.

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 3 & 0 \\ 0 & 3 & 5 \end{bmatrix}$$

4. For this problem we will attempt to find the Least Squares approximation to a line $y = a_0 + a_1 x$ for the points $(0,2)$, $(1,3)$, $(2,7)$ and $(3,6)$.

   (a) Graph the points and guess what the function may be?

   (b) My guess is $y = x + 3$ (not a very good guess). Compute the error for my function and yours, which is better? Use

   $$E(f) = \sum (y_i - f(x_i))^2$$

   as we did in class.

   (c) What are the matrices A, $\mathbf{u}$ and $\mathbf{y}$ to get

   $$A\mathbf{u} = \mathbf{y}.$$

   (d) Compute

   $$\mathbf{u} = (A^T A)^{-1} A^T \mathbf{y}.$$

   (e) What is the least squares function $f(x) = a_0 + a_1 x$.

5. For this problem we will attempt to find the Least Squares approximation to a quadratic $y = a_0 + a_1 x + a_2 x^2$ for the points $(0, 2)$, $(1, 3)$, $(2, 7)$ and $(3, 6)$.

   (a) Graph the points and guess what the function may be?

   (b) My guess is $y = x^2 + 1$ (not a very good guess). Compute the error for my function and yours, which is better? Use

   $$E(f) = \sum (y_i - f(x_i))^2$$

   as we did in class (hint my error is 22).

   (c) What are the matrices A, $\mathbf{u}$ and $\mathbf{y}$ to get

   $$A\mathbf{u} = \mathbf{y}.$$

   (d) Compute

   $$\mathbf{u} = (A^T A)^{-1} A^T \mathbf{y}.$$

   (e) What is the least squares function $f(x) = a_0 + a_1 x + a_2 x^2$.

6. Find the SVD for the following matrix.

   $$A = \begin{bmatrix} 1 & 4 \\ 0 & 3 \\ 0 & 3 \end{bmatrix}$$