

Math 3520 - Test 2 Review

1 Cardinality

1. Define cardinality. That is $A \sim B$ if and only if \dots
2. Know \sim is an equivalence relation and what that means.
3. Let $A = [0, 2]$ and $B = [-1, 6]$. Show $f : A \rightarrow B$ given by $f(x) = \frac{7}{2}x - 1$ is a bijection. What does this fact tell us about the sets A and B?
4. Show the sets have the same cardinality
 - (a) $\mathbb{N} \sim 2\mathbb{N}$
 - (b) \mathbb{N} and \mathbb{Z}
 - (c) \mathbb{N} and \mathbb{Q}
 - (d) $\{a, b, c\}$ and $\{1, 2, 7\}$
 - (e) $\mathbb{N} \times \{1, 2, 3, 4\}$ and \mathbb{N}
 - (f) $[0, 1)$ and $(2, 3]$
 - (g) $(0, 1)$ and \mathbb{R}
5. Show the sets **do not** have the same cardinality
 - (a) \mathbb{N} and \mathbb{R} .
 - (b) \mathbb{N} and the set of irrationals.

2 Number Theory

6. Let $a, b, c \in \mathbb{Z}$ with $b \neq 0$. Show if $a|b$ and $b|c$ then $a|c$.
7. Let $a, b, c, x, y \in \mathbb{Z}$. Show if $a|b$ and $a|c$ then $a|bx + cy$.
8. Let $a, b, c, d \in \mathbb{Z}$. Show if $a|b$ and $c|d$ then $ac|ad + bc$.
9. Prove $3|n^3 - n$ for every integer n .
10. Let $a, b, c \in \mathbb{Z}$. If $a|bc$ and $\gcd(a, b) = 1$ then $a|c$.
11. Let $a, b \in \mathbb{Z}$ and let p be a prime. If $p|ab$ then $p|a$ or $p|b$.
12. Let $a, b, c \in \mathbb{Z}$ where $\gcd(a, b) = 1$. If $a|c$ and $b|c$ then $ab|c$.

13. Let $n \in \mathbb{Z}$ be odd. Prove $n^2 \equiv 1 \pmod{4}$.
14. Let $n \in \mathbb{Z}$ be odd. Prove $n^2 \equiv 1 \pmod{8}$.
15. Let $a, b, c \in \mathbb{Z}$ so that $a^2 + b^2 = c^2$. Prove $4|ab$ (hint use 14).
16. Prove or disprove if $a, b \in \mathbb{Z}$ and a, b are odd then $8|a^2 - b^2$.
17. Prove $\sqrt{3}$ is irrational.
18. For the following pairs of numbers, find their gcd and find a linear combination of the numbers equal to their gcd.
 - (a) $a = 78$ and $b = 48$
 - (b) $a = 79$ and $b = 49$
 - (c) $a = 253$ and $b = 207$

3 Group Theory

19. Definition of a **subgroup** and of an **isomorphism**.
20. For the group $(\mathbb{Z}, +)$ Show $H = \{2n + 1 : n \in \mathbb{Z}\}$ is not a subgroup.
21. State the definition of an **Algebraic structure** and a **binary operation**.
22. State the definition of a **group**.
23. State the definition of an **Abelian group**.
24. State and prove the cancellation law.
25. Are the following algebraic structures groups? If it is a group prove it. If not prove it is not a group.
 - (a) (\mathbb{Q}^*, \cdot) where \cdot is regular multiplication.
 - (b) (\mathbb{Z}, \odot) where $a \odot b = a + b - 2$.
 - (c) (\mathbb{Z}, \otimes) where $a \otimes b = ab + a + b$.
 - (d) $(\mathbb{Q} \setminus \{-1\}, \otimes)$ where $a \otimes b = ab + a + b$.
26. For the following algebraic structures write the operation tables. State if the structure is a group or not. If not state which property fails and how. Identify the identity in each case.

- (a) $(\mathbb{Z}_5, +)$
- (b) (\mathbb{Z}_5, \cdot)
- (c) (\mathbb{Z}_6^*, \cdot)
- (d) (S_3, \circ)
- (e) (D, \circ) where

$$D = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\}$$

- (f) (D, \circ) where $D \subseteq S_3$ defined as

$$D = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\}$$

- (g) (M, \cdot) where $M \subseteq GL_2(\mathbb{R})$ defined as

$$M = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$$

27. We showed in class that some groups are “the same” by finding a function from one group to the other that preserved the operation. The following pairs of groups are “the same” in that sense find the function and verify that it preserves the operation.

- (a) $(\mathbb{Z}_4, +)$ and (\mathbb{Z}_5^*, \cdot)
- (b) $(\mathbb{Z}_2, +)$ and (M, \cdot) where $M \subseteq GL_2(\mathbb{R})$ defined as

$$M = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

- (c) $(\mathbb{Z}_{12}^*, \cdot)$ and $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$

28. Let G be a group prove

- (a) The identity in G is unique.
- (b) If $a \in G$ then a^{-1} is unique.
- (c) If $a \in G$ then $(a^{-1})^{-1} = a$.
- (d) If $a, b \in G$ then $(ab)^{-1} = b^{-1}a^{-1}$.

29. Let G be a group prove and let $a, b \in G$. If a and b commute then a^{-1} and b^{-1} commute.

30. Let $(G, *)$ be a group prove and let $a, b, c \in G$. Prove the following equations have an unique solution $x \in G$.

- $a * x * c = b$
- $c * a * x = b$

31. Let $(G, *)$ be a group prove.

Assume there is some element $f \in G$ so that for some element $a \in G$ we have $af = a$. Is f the identity? Prove or disprove.

32. Let $(G, *)$ be a group prove.

If $a * a = e$ for all $a \in G$ then G is Abelian.

33. For the group $(\mathbb{Z}, +)$ prove, using the 2-step subspace test, that $H = \{3n : n \in \mathbb{Z}\}$ is a subgroup.

34. For the group (S_3, \circ) Show

$$H = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \right\}$$

is not a subgroup of S_3 .

35. Let G be any group and let g be a fixed element of G then prove, using the 2-step subspace test, that $H = \{gag^{-1} | a \in G\}$ is a subgroup.

36. Let G be any group then prove, using the 2-step subspace test, that $H = \{a \in G | ag = ga \forall g \in G\}$ is a subgroup.

37. For the groups $G_1 = (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, +)$ and $G_2 = (\mathbb{Z}_8, +)$

- (a) Find the orders of the elements $(1, 1, 1)$ and $(1, 0, 1)$ in G_1 and the orders of the elements 1, 2, 3 in G_2 ?
- (b) Are any of the above elements generators for their respective groups?
- (c) Why aren't the groups isomorphic?

38. For the groups $G_1 = (\mathbb{Z}_9^*, +)$ and $G_2 = (\mathbb{Z}_6, +)$

- (a) Find the orders of three different non identity elements in G_1 and the orders of the elements 1, 2, 3 in G_2 ?
- (b) Are any of the above elements generators for their respective groups?

- (c) The two groups are isomorphic. Find the isomorphism $f : G_1 \rightarrow G_2$.
39. Note that (G, \cdot) is a group where $G = \{2^n : n \in \mathbb{Z}\}$ and \cdot is regular multiplication. Prove Axioms G_1 and G_3 for (G, \cdot) .
40. Note that (G, \cdot) is a group where $G = \{2^n : n \in \mathbb{Z}\}$ and \cdot is regular multiplication. Show $G \cong \mathbb{Z}$ where \mathbb{Z} is a group over addition. I used the isomorphism $f : \mathbb{Z} \rightarrow G$. We need to prove f preserves the operation and that f is a bijection.

4 New Stuff

41. Define $\lim_{x \rightarrow c} f(x) = L$
42. Compute and prove the following limits
- (a) $\lim_{x \rightarrow 2} 3x + 2$
 - (b) $\lim_{x \rightarrow -3} x^2 + 1$
 - (c) $\lim_{x \rightarrow -1} 2x^2$
 - (d) $\lim_{x \rightarrow 2} x^3$
 - (e) $\lim_{x \rightarrow -2} x^3 + 1$