MATH 5320 Test 2: Practice

1 Continuity

1. Show $f(x) = x^2 + 1$ is continuous at $x = 3$ (using the $\varepsilon - \delta$ definition).
2. Show $f(x) = x^2 + 1$ is continuous where $f : \mathbb{R} \to \mathbb{R}$ (using the $\varepsilon - \delta$ definition).
3. Show $f(x) = x^2 + 1$ is uniformly continuous at $f : [-10, 7] \to \mathbb{R}$ (using the $\varepsilon - \delta$ definition).
4. State the IVT and the EVT.
5. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and let $\text{range}(f) \subseteq \mathbb{Q}$. Prove $f(x)$ is constant.
6. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and injective. Prove $f(x)$ is strictly monotone.

2 Differentiability

7. Compute the derivatives of the following using the definition:
   (a) $f(x) = x^2$
   (b) $f(x) = \begin{cases} 3x - 2 + \frac{x^2}{|x|} & : x \neq 0 \\ -2 & : x = 0 \end{cases}$ at $x = 0$.
   (c) $f(x) = \begin{cases} x \sin(\frac{1}{x}) & : x \neq 0 \\ 0 & : x = 0 \end{cases}$ at $x = 0$.
   (d) $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & : x \neq 0 \\ 0 & : x = 0 \end{cases}$ at $x = 0$.
   (e) $f(x) = \begin{cases} \sin(\frac{1}{x}) & : x \neq 0 \\ 0 & : x = 0 \end{cases}$ at $x = 0$.
8. Prove if $f(x)$ and $g(x)$ are differentiable at $x = c$ then $[f(c)g(c)]' = f'(c)g(c) + f(c)g'(c)$.
9. Use the product rule to show

$$[f(x)g(x)h(x)]' = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

10. Prove if f is differentiable at $x = c$ then f is continuous at $x = c$.

11. Prove the following function is continuous but not differentiable. $f(x) = \begin{cases} x^2 & : x < 1 \\ 3x - 2 & : x \geq 1 \end{cases}$.

12. Prove if $|f(x)| \leq x^2$ for all $x \in \mathbb{R}$ then $f'(0) = 0$.

13. State the MVT.

14. Find all values of c from the MVT for the following

    (a) $f(x) = 3x^2 + 5x + 7$; $[1, 7]$

    (b) $f(x) = 3x^2 + 5x + 7$; $[a, b]$

    (c) $f(x) = |x|$; $[1, 7]$

    (d) $f(x) = |x|$; $[-1, 7]$

15. Prove If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and there is some $M \in R$ so that $|f'(x)| \leq M$ for all $x \in \mathbb{R}$ then f is uniformly continuous. Hint: I used the MVT.

16. State Taylor's Theorem

17. Use Taylor's Theorem (n=3) to find a polynomial to approximate the following functions at $a = 0$. Bound the remainder term for values in the interval $[0, 1]$.

    (a) $f(x) = \sin(2x)$.

    (b) $f(x) = \cos(3x)$.

    (c) $f(x) = e^{5x}$.

18. Let $f : \mathbb{R} \to \mathbb{R}$ have first and second derivatives. If $f(0) = 0$, $f'(0) = 0$ and $f''(x) > 2$ for all $x \in \mathbb{R}$ then $f(x) > x^2$ for all $x > 0$. Hint: I used the Taylor's Theorem.

3 Integration

19. Prove (using Riemann Sums) that

$$f(x) = \begin{cases} 2 & : x > 2 \\ -3 & : x \leq 2 \end{cases}$$

is integrable over the interval $[0, 3]$. What is that integral?

20. Let $f(x) = x^2+1$, $[a,b] = [1,4]$ and let $\mathcal{P} = \{1,2,3,3.5,3.7,4\}$ be a artition for [1,4] and let $\mathcal{S} = \{1, 2.2, 3.1, 3.6, 4\}$ be a sampling. Compute $RS(f,\mathcal{P},\mathcal{S})$, $US(f,\mathcal{P})$ and $LS(f,\mathcal{P})$.

21. Assume $f : [a,b] \to \mathbb{R}$ is integrable. Show if $f(x) \geq 0$ then $\int_a^b f(x) \geq 0$. Prove using the definition of the integral.

22. Assume $f : [a,b] \to \mathbb{R}$ is integrable. Show if $\int_a^b f(x) = 0$ and $0 \leq g(x) \leq f(x)$ then $g(x)$ is integrable and $\int_a^b g(x) = 0$. Prove using the definition of the integral.

23. ~~Assume $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are integrable and let $k \in \mathbb{R}$. Prove~~

    (a) ~~The function $kf(x)$ is integrable and that $\int_a^b kf(x) = k\int_a^b f(x)$.~~

    (b) ~~The function $f(x)+g(x)$ is integrable and that $\int_a^b f(x)+g(x) = \int_a^b f(x) + \int_a^b g(x)$.~~

24. State the Box Sums Criteria (the BSC).

25. Prove if $f(x)$ is continuous then $f(x)$ is integrable (use the BS Criteria).

26. Prove if $f(x)$ is monotone then $f(x)$ is integrable (use the BS Criteria).

27. 5.1:1,2,3,4,7*,14

28. ~~5.2:1,2,3,6~~

29. 5.3:4,5,6

30. 5.4:9, 10

31. State the FTC v0, the FTC v1, the FTC v2 and the MVTI.

32. Let $f_n(x) = x^n$ where $n \in \mathbb{N}$.

    (a) Graph $f_n(x)$ on the interval $[0,1]$ for several values of n until you see the pattern. Explain the pattern.

    (b) Compute $\int_0^1 f_n$.

    (c) Find the limit $\lim_{n\to\infty} \int_0^1 f_n$.

    (d) What does the MVTI say about

        i. $f(x) = x^2+1$ over $[a,b] = [-1,3]$.

        ii. $f(x) = |x|$ over $[a,b] = [-3,3]$.