1 Sups

1. Definitions of sup, inf, upper bound, lower bound, bounded away
2. Prove If the sup(A) and the sup(B) exist then $\sup(A) + \sup(B) = \sup(A + B)$.
3. Prove If the sup(A) exists then $\sup(A) = -\inf(-A)$.
4. The Lemma - its statement and proof
5. Let $S = \{\frac{1}{1+x^2} : x \in \mathbb{R}\}$. Prove or disprove S is bounded away from zero.
6. Let $S = \{1 + x^2 : x \in \mathbb{R}\}$. Prove or disprove S is bounded away from zero.
7. Prove $\mathbb{N}$ is unbounded
8. Prove (and be able to state) squeeze in.
9. Let $x \in \mathbb{R}$ so that $x \geq 0$, Prove if for all $\varepsilon > 0$ we have $x < \varepsilon$ then $x = 0$.

2 Limits

10. Definitions of $a_n \to a$, $a_n \to \infty$, $a_n \to -\infty$
11. Prove the following:
    (a) $\lim_{n\to\infty} k = k$.
    (b) $\lim_{n\to\infty} ka_n = ka$.
    (c) $\lim_{n\to\infty} a_n + b_n = a + b$.
    (d) $\lim_{n\to\infty} a_n b_n = ab$.
    (e) If a_n converges then a_n is bounded.
    (f) If a_n converges then the limit is unique.
    (g) If $a_n \to 0$ and (b_n) is bounded then $a_n b_n \to 0$.
    (h) If (a_n) is bounded then $(\frac{1}{a_n})$ is bounded away from zero.
    (i) If (a_n) is bounded away from zero then $(\frac{1}{a_n})$ is bounded.
12. Prove (use the $\epsilon = N$ definition).

(a) $\lim_{n\to\infty} \frac{n}{n+3} = 1$

(b) $\lim_{n\to\infty} \frac{n^2+sin(n)}{n^2+3-cos(n)} = 1$

(c) $\lim_{n\to\infty} \frac{n+4}{n^2+3} = 0$

(d) $\lim_{n\to\infty} \sqrt{n} = \infty$

13. For the following questions use

$$a_1 = 6, \text{ and } a_n = \sqrt{3 + a_{n-1}}$$

(a) Prove (a_n) is monotone.

(b) Prove (a_n) is bounded.

(c) Use the MCT (and state the MCT) to prove (a_n) converges.

(d) What is the limit?

14. For the following questions use

$$a_1 = 3, \text{ and } a_n = 1 + \frac{1}{2 + \frac{1}{1+a_{n-1}}}$$

(a) Prove (a_n) is monotone.

(b) Prove (a_n) is bounded.

(c) Use the MCT (and state the MCT) to prove (a_n) converges.

(d) What is the limit?

3 Subsequences

15. State the Bolzano-Weiersrtass Theorem.

16. ~~Let (q_n) be an enumeration of the rationals. Prove that there is a subsequence of (q_n) that converges to 3.~~

4 Limits of functions

17. State the SCL

18. State and Prove the Squeeze Theorem.

19. Find the limit and prove it for the following using the $\varepsilon - \delta$ definition for the limit

   (a) $\lim_{x \to 3} x^2 + 2$
   (b) $\lim_{x \to -2} \frac{x}{x+11}$
   (c) $\lim_{x \to \infty} \ln(x)$

5 Continuity

20. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 - 2$. Prove

   (a) f is continuous at $x = -2$ from the definition
   (b) f is continuous from the definition
   (c) f is **not** uniformly continuous.

21. Define $f : [-20, 20] \to \mathbb{R}$ by $f(x) = x^2 - 2$. Prove

   (a) f is continuous at $x = -2$ from the definition
   (b) f is continuous from the definition
   (c) f is uniformly continuous.

22. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy

   - $f(x + y) = f(x) + f(y)$, and
   - $f(kx) = kf(x)$

   for any $k \in \mathbb{R}$ and $x, y \in \mathbb{R}$. Prove

   (a) f is continuous from the definition
   (b) f is uniformly continuous.