ICPS Fall 2015, December 5th and December 12th

1 What is an infinity?

1.1 How to Count

Olivia

1 → candy bar
2 → tootsie roll
3 → lolly pop
3 → chocolate

Emma

1 → candy bar
2 → tootsie roll
3 → tootsie roll

Uncle Frank

1 → candy bar
2 → tootsie roll
lolly pop

Jada

1 → candy bar
2 → tootsie roll
3 → lolly pop

Who counted correctly?

1.2 Functions

Some reminders

Natural Numbers	$\mathbb{N}$	$= \{1, 2, 3, \dots\}$
Integers	$\mathbb{Z}$	$= \{\dots, -2, -1, 0, 1, 2, 3, \dots\}$
Rationals	$\mathbb{Q}$	$= \{\frac{m}{n} : m, n \in \mathbb{Z}\}$
Real Numbers	$\mathbb{R}$	= anything that can be written as a decimal

Back from algebra class we have functions presented as

$$f: \text{ some set } \rightarrow \text{ Another Set, defined by the formula } f(x) = \text{ some rule}$$

Such as $f : \mathbb{R} \rightarrow \mathbb{R}$ where $f(x) = x^2$.

We have names for the sets. What are the names you remember from algebra class? Let $f : A \rightarrow B$ be a function we call the set A the domain of f and the set B the codomain of f.

And you mentioned one more name what is the range of f?

Domain	A	the x values
Codomain	B	the possible y values
Range	$f(A)$	the y values

Back to our example $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$ (see Figure 1). What are the domain, codomain and range of f?

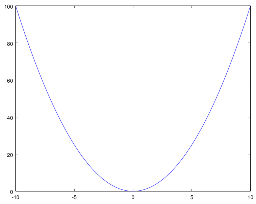

Figure 1: $y = x^2$

I get Domain$(f) = \mathbb{R}$, Codomain$(f) = \mathbb{R}$ and Range$(f) = [0, \infty)$.

Problem 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be given by the formula $f(x) = 4 - x^2$ (see Figure 2). What are the domain, codomain and range of f?*

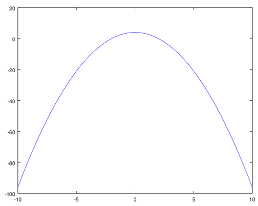

Figure 2: $y = 4 - x^2$

Problem 1.2. *Let $f : [-1, \infty] \to \mathbb{R}$ be given by the formula $f(x) = 5 - x$ (see Figure 3). What are the domain, codomain and range of f?*

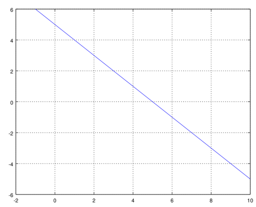

Figure 3: $y = 5 - x$

Problem 1.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be given by the formula $f(x) = \sin(x)$. What are the domain, codomain and range of f?*

Problem 1.4. *Let $f : [-2, 2] \to \mathbb{R}$ be given by the formula $f(x) = \sqrt{4 - x^2}$. What are the domain, codomain and range of f?*

We've seen functions from one set to another, but it was always $\mathbb{R}$ or some interval in $\mathbb{R}$. But those sets can be any set.

Problem 1.5. *Find the the domain, codomain and range of each f defined below.*

1. *Let $f : \mathbb{N} \to \mathbb{Z}$ be given by the formula $f(n) = 2n$.*
2. *Let $f : \mathbb{Z} \to \mathbb{N}$ be given by the formula*

$$f(n) = \begin{cases} n/2 & : \text{ n is even} \\ (1-n)/2 & : \text{ n is odd} \end{cases}$$

3. *Let $f : \{1, 2, 3\} \to \{a, b, c\}$ given by*

$$f(1) = a, f(2) = c \text{ and } f(3) = a$$

4. *Let f be the function that goes from the set of all rectangles to $\mathbb{R}$ where $f($ a rectangle $)$ = the area if that rectangle.*
5. *Define $A = \{1, 2, 3\}$. Let $f : \mathcal{P}(A) \to A$ be given by the rule $f(S) =$ minimum of the set S. Recall $\mathcal{P}(A)$ is called the power set of A. It is all possible subsets of A, so*

$$\mathcal{P}(A) = \big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\big\}.$$

1.3 Properties of Functions

Injective or one to one

Note that the function in Figure 4 is bf NOT one to one. However, we say the

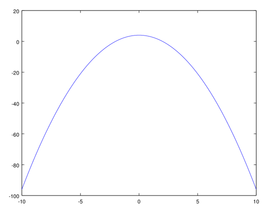

Figure 4: $y = 4 - x^2$

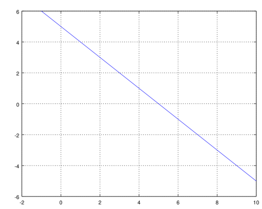

Figure 5: $y = 5 - x$

function in Figure 5 is one to one.

What does one to one mean? Graphically? We say a function is **one to one** if each y in the range occurs exactly once.

Surjective or onto

- $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is bf NOT onto.
- $f : \mathbb{R} \to [0, \infty)$ given by $f(x) = x^2$ is onto.

Let's graph the two and see what we get. Doe anything stand out in the graphs? We say a function is **onto** if range equals the codomain.

Problem 1.6. *For the following functions determine if the function is one-to-one, onto, both or neither.*

1. *Let* $f : \{1, 2, 3\} \to \{a, b, c\}$ *given by*

$$f(1) = a, f(2) = c \text{ and } f(3) = a$$

2. *Let* $f : \{1, 2, 3\} \to \{a, b, c\}$ *given by*

$$f(1) = b, f(2) = c \text{ and } f(3) = a$$

3. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be given by the formula* $f(x) = 2x$.
4. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be given by the formula* $f(x) = x^2$.
5. *Let* $f : \mathbb{N} \to \mathbb{Z}$ *be given by the formula* $f(n) = 2n$.
6. *Let* $f : \mathbb{Z} \to \mathbb{N}$ *be given by the formula*

$$f(n) = \begin{cases} n/2 & : n \text{ is even} \\ (1-n)/2 & : n \text{ is odd} \end{cases}$$

7. *Let* f *be the function that goes from the set of all rectangles to* $\mathbb{R}$ *where* $f($ *a rectangle* $)$ = *the area if that rectangle.*

What did my nieces do with their candy...

1.4 Cardinality

The count of elements in a set is the cardinality of the set. This is a fairly easy concept for sets like $A = \{a, b, c\}$. The cardinality of the set is 3 and it is written as $|A| = 3$. But how do I work with infinite sets?

We need a slightly more rigorous definition for cardinality. Let's use what my nieces used. The sets A and B have the same cardinality if there is a one-to-one and onto function $f : A \to B$.

So $\{a, b, c\}$ has count 3 since the function $f : \{1, 2, 3\} \to \{a, b, c\}$ given by

$$f(1) = b, f(2) = c \text{ and } f(3) = a$$

is one-to-one and onto. We will denote as $|\{1, 2, 3\}| = |\{a, b, c\}|$.

We can also say things like $|A| < |B|$ or $|A| \neq |B|$.

Problem 1.7. *Recall* $f : \mathbb{Z} \to \mathbb{N}$ *be given by the formula*

$$f(n) = \begin{cases} n/2 & : n \text{ is even} \\ (1-n)/2 & : n \text{ is odd} \end{cases}$$

is one-to-one and onto. What does this say about the sets $\mathbb{Z}$ *and* $\mathbb{N}$*?*

Problem 1.8. *Recall* $f : \mathbb{N} \to 2\mathbb{N}$ *be given by the formula* $f(n) = 2n$. *Is* f *one-to-one and onto? What does this say about the sets* $\mathbb{N}$ *and* $2\mathbb{N}$*?*

1.5 Hilbert's Hotel

David Hilbert opened up a very special hotel. In his hotel their were infinitely many rooms. The rooms were numbered $1, 2, 3, 4, \ldots$ His hotel was very popular. IN fact it was so popular every room was filled.

So he hung up a sign "No Vacancy. But plenty of room"

A traveler came by and said he would like to rent a room for the night. He asked "What does that sign mean?" Hilbert said, "Every room is filled. However, we do have a room for you." "How?" asked the traveler. Hilbert said, "You take room one and tell the person in room one to move room two." The traveler says, "Bu t then the person in room 2 will not have a room!" "The person from room 2 should go to room 3 and ask the person in room 3 to move 4 and so on and so on. That way every person will have a room for the night. You see the person in room 500 will move to room 501 and so on."

What should Hilbert do when

1. 100 people arrive to the completely filled hotel
2. a long bus with infinitely many people arrive to the completely filled hotel
3. infinitely many long buses with infinitely many people arrive to the completely filled hotel

Note $|\mathbb{N}| = |\mathbb{Z}|$ and $|\mathbb{N}| = |\mathbb{Q}|$. How?

1.6 The Barber of Seville

The barber of Seville shaves every man in Seville who does not shave himself. Who shaves the barber?

Assume $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ is on to one and onto. So we are assuming $|\mathbb{N}| = |\mathcal{P}(\mathbb{N})|$.

Define the set S by

$$S = \{n \in \mathbb{N} : n \notin f(n)\}.$$

Since f is onto and since $S \in \mathcal{P}(\mathbb{N})$ we have that there exists some $m \in \mathbb{N}$ so that $f(m) = S$.

Question: Is $m \in S$?

So $|\mathbb{N}| \neq |\mathcal{P}(\mathbb{N})|$!

2 Areas formed by bicycle tires and toy ducks? (or My duck Newton)

2.1 My duck Newton

2.2 Area of an Ikon

2.3 Mamikon's Theorem

2.4 Back to the duck (he will not be denied)

2.5 Some problems for y'all

3 Circles in Circles

3.1 Crayfish or Maths

3.2 Descartes Theorem

3.3 Soddy Circles

3.4 Ford Circles