2.6

① $f(x) = 3e^{-3x} \quad x > 0$

(a) • $E(x) = \int_0^\infty x\, f(x)\, dx = \int x\, 3e^{-3x}\, dx$

x	$+$	$3e^{-3x}$
1	$-$	$-e^{-3x}$
0		$\frac{1}{3}e^{-3x}$

$$= x(-e^{-3x}) - (1)\left(\tfrac{1}{3}e^{-3x}\right)\Big|_0^\infty$$

$$= \underset{x=\infty}{(0 - 0)} - \underset{x=0}{\left(0 - \left(\tfrac{1}{3}e^{0}\right)\right)} = \boxed{\tfrac{1}{3}}$$

• $E(x^2) = 3\int_0^\infty x^2 e^{-3x}\, dx =$

x^2	$+$	e^{-3x}
$2x$	$-$	$-\frac{1}{3}e^{-3x}$
2	$+$	$\frac{1}{9}e^{-3x}$
0		$-\frac{1}{27}e^{-3x}$

$$= 3\left[x^2\left(-\tfrac{1}{3}e^{-3x}\right) - 2x\left(\tfrac{1}{9}e^{-3x}\right) + 2\left(-\tfrac{1}{27}e^{-3x}\right)\right]$$

$$= -e^{-3x}\left[x^2 + \tfrac{2}{3}x + \tfrac{2}{9}\right]\Big|_0^\infty$$

$$= \underset{x=\infty}{0} - \underset{x=0}{-e^{0}\left[0 + 0 + \tfrac{2}{9}\right]} = \boxed{\tfrac{2}{9}}$$

• $E(x^3) = \int_0^\infty x^3 \cdot 3e^{-3x}\, dx$

x^3	$+$	$3e^{-3x}$
$3x^2$	$-$	$-e^{-3x}$
$6x$	$+$	$\frac{1}{3}e^{-3x}$
6	$-$	$-\frac{1}{9}e^{-3x}$
0		$\frac{1}{27}e^{-3x}$

$$= \left[-x^3e^{-3x} - 3x^2\left(\tfrac{1}{3}e^{-3x}\right) - 6x\left(\tfrac{1}{9}e^{-3x}\right) - 6\left(\tfrac{1}{27}e^{-3x}\right)\right]\Big|_0^\infty$$

$$= -e^{-3x}\left[x^3 + x^2 + \tfrac{2}{3}x + \tfrac{2}{9}\right]\Big|_0^\infty$$

$$= 0 - -e^{0}\left[0 + 0 + 0 + \tfrac{2}{9}\right] = 2/9$$

(b) $\mu = E(x) = \frac{1}{3}$

$$VAR(x) = E(x^2) - E(x)^2 = \tfrac{2}{9} - \left(\tfrac{1}{3}\right)^2 = \boxed{\tfrac{1}{9}}$$

(2) $f(x) = c(x^3+1) \quad : \quad 0 \le x \le 2$

(a) To find c NOTE $\int f(x)\,dx = 1$. So

$$\int_0^2 c(x^3+1)\,dx = c\left[\frac{x^4}{4} + x\right]\Big|_0^2 = c\left[\frac{16}{4}+2\right] - 0$$

$$= 6c = 1$$

So $c = \frac{1}{6}$

& $f(x) = \frac{1}{6}[x^3+1]$

(b) $P(1 < x < 2) = \int_1^2 f(x)\,dx$

$$= \int_1^2 \frac{1}{6}[x^3+1]\,dx = \frac{1}{6}\left[\frac{x^4}{4}+x\right]\Big|_1^2$$

$$= \frac{1}{6}\left[\frac{2^4}{4}+2\right] - \frac{1}{6}\left[\frac{1}{4}+1\right] = 1 - \frac{1}{6}\left[\frac{5}{4}\right] = \frac{24-5}{24}$$

$$= \boxed{19/24}$$

(c) $E(x) = \int_0^2 x\, f(x)\,dx$

$$= \int_0^2 x \, \frac{1}{6}[x^3+1]\,dx = \frac{1}{6}\int_0^2 x^4 + x\,dx = \frac{1}{6}\left[\frac{x^5}{5}+\frac{x^2}{2}\right]_0^2$$

$$= \frac{1}{6}\left[\frac{32}{5} - \frac{\overset{2}{\cancel{4}}}{\cancel{2}}\right] - \frac{1}{6}\left[0+0\right] = \frac{1}{6}\cdot\frac{32-10}{5} = \frac{22}{30} = \boxed{\frac{11}{15}}$$

$E(x^2) = \int_0^2 x^2 f(x)\,dx$

$$= \int_0^2 x^2 \cdot \frac{1}{6}[x^3+1]\,dx = \frac{1}{6}\int x^5 + x^2\,dx = \frac{1}{6}\left[\frac{x^6}{6}+\frac{x^3}{3}\right]_0^2$$

$$\frac{1}{6}\left[\frac{\overset{32}{\cancel{64}}}{\underset{3}{\cancel{6}}} + \frac{8}{3}\right] - 0 = \frac{40}{18} = \frac{20}{9}$$

(2.) (d) $\mu = E(x) = \frac{11}{15}$

& $VAR(x) = E(x^2) - E(x)^2$

$= \frac{20}{9} - \left(\frac{11}{15}\right)^2$

(3.)

X	-1	0	1	2
$f(x)$	0.2	0.4	0.3	0.1

(a) $\boxed{c = 0.4}$

X	-1	0	1	2	
$f(x)$	$\frac{2}{10}$	$\frac{4}{10}$	$\frac{3}{10}$	$\frac{1}{10}$	
$xf(x)$	$\frac{-2}{10}$	0	$\frac{3}{10}$	$\frac{2}{10}$	$\underset{sum}{\Sigma} \to \frac{3}{10}$
$x^2 f(x)$	$\frac{2}{10}$	0	$\frac{3}{10}$	$\frac{8}{10}$	$\underset{sum}{\Sigma} \to \frac{13}{10}$

(b) $P(1 < x < 2) = 0$

(c) $E(x) = \frac{3}{10}$

$E(x^2) = \frac{13}{10}$

(d) $VAR(x) = E(x^2) - E(x)^2 = \frac{13}{10} - \left(\frac{3}{10}\right)^2 = \frac{130-9}{100} = \boxed{\frac{121}{100}}$

$\mu = E(x) = \boxed{\frac{3}{10}}$

(4) MOMENT GENERATING FUNCTION FOR $f(x) = 3e^{-3x} \quad x > 0$

$$M_X(t) = E(e^{tX})$$

$$= \int_0^\infty e^{tx} \cdot 3e^{-3x}\, dx = 3\int_0^\infty e^{(t-3)x}\, dx$$

$$= 3 \frac{1}{t-3} e^{(t-3)x} \Big|_0^\infty = 0 - 3 \frac{1}{t-3} e^0 = \boxed{\frac{3}{3-t}}$$

$t < 3$

(5) FIND $M_X(t)$ for

X	0	1	2
$f(x)$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{5}{10}$
$e^{tx} \cdot f(x)$	$\frac{2}{10}$	$e^t \frac{3}{10}$	$\frac{5}{10} e^{2t}$

$\longrightarrow$ Sum $= \frac{2}{10} + \frac{3}{10} e^t + \frac{5}{10} e^{2t}$

$X = 1$ so

$e^{tx} f(x) = e^t f(1) = e^t \left(\frac{3}{10}\right)$

$X = 0$ so

$e^{tx} f(x) = e^0 f(0) = 1 \cdot \frac{2}{10}$

$$\boxed{M_X(t) = \frac{2}{10} + \frac{3}{10} e^t + \frac{5}{10} e^{2t}}$$

6. Let $M_X(t) = \frac{1}{1-t}$

$M_X'(0) = E(X^1)$

$E(X) = 1$

$M_X'(t) = (1-t)^{-2}$ | $M_X'(0) = (1-0)^{-2} = 1$

$M_X''(t) = 2(1-t)^{-3}$ | $M_X''(0) = 2(1-0)^{-3} = 2$

$E(X^2) = 2$

Recall

$M_X^{(n)}(0) = E(X^n)$

So $\mu = E(X) = 1$

AND

$$\begin{aligned} VAR(X) &= E(X^2) - E(X)^2 \\ &= 2 - (1)^2 = 1 \end{aligned}$$

So $\mu = 1$ AND $\sigma = \sqrt{VAR(X)} = \sqrt{1} = 1$