Math 6250: Test 2

Name:____

1. For the following sets in $(\mathbb{R}, e)$, the reals with the usual metric, compute the boundary, the closure and the interior of the given set. Also determine if the set is open, closed, both or neither.

   (a) $A = \mathbb{Q}$

   (b) $B = (0, 1)$

   (c) $C = [-2, 17]$

   (d) $D = \{\frac{1}{n} : n \in \mathbb{N}\}$

   (e) $E = [0, \infty) \setminus \mathbb{N}$

   (f) The Cantor set

2. Consider the following metric spaces. Determine if the metric space is complete. An explanation of your answer is necessary.

   (a) $\mathbb{R}$ with usual metric

   (b) $\mathbb{Q}$ with usual metric

   (c) $\mathbb{N}$ with metric $d(n, m) = |\frac{1}{n} - \frac{1}{m}|$

   (d) $\widetilde{\mathbb{N}}$ as in class.

   (e) ~~$X = \{\frac{1}{n} : n \in \mathbb{N}\}$~~

   (f) $C[0, 1]$ with the metric $d_1(f, g) = \int_0^1 |f - g|dx$

   (g) $C[0, 1]$ with the metric $d_\infty(f, g) = \max |f(x) - g(x)|$

3. Prove the following: Let (X, d) be a complete metric space so that $diam(X) < \infty$. Assume $f : X \to X$ is contractive. Show that there exists a unique point $x \in X$ so that $f(x) = x$. We will call this point the fixed point of f in X.

4. Consider $([0, \infty), d)$ the non-negative real numbers with the usual metric. Find a function $f : X \to X$ where f is contractive and does not have a fixed point. Explain your answer.

5. Consider the function $f : [0, 1] \to [0, 1]$ where $f(x) = 1 - x^3$.

   (a) Is f contractive? Prove or disprove.

   (b) Does f have a fixed point? If so find it. If not provide a proof.

6. Find a function $f : (0,1] \to (0,1]$ where f is continuous and f does not have fixed point.

7. Consider $(\mathbb{N}, d)$ where $d(n,m) = |\frac{1}{n} - \frac{1}{m}|$ as previously discussed. Find a function $f : X \to X$ where f is contractive and does not have a fixed point. Explain your answer.

8. Let $X = C^{\infty}[0,1]$ be the functions with infinitely many continuous derivatives with metric

$$d(f,g) = \int_0^1 |f-g|dx.$$

   (a) Define $D : X \to X$ by $D(f(x)) = f'(x)$. Does D have a nonzero fixed point? If so find it.

   (b) Define $D : X \to X$ by $D(f(x)) = f''(x) + 2f(x)$. Does D have a nonzero fixed point? If so find it.