**Math 6250: Test 1**

**Name:**____________________________________________

1. Define the following terms.

   (a) $(X, d)$ is a metric space $\Longleftrightarrow$

   (b) $\operatorname{diam}(S)$ in the metric space $(X, d)$ $\Longleftrightarrow$

   (c) $\operatorname{dist}(x, S)$ in the metric space $(X, d)$ $\Longleftrightarrow$

2. Some general questions.

   (a) Compute the $\inf\{\frac{1}{2} - \frac{1}{n+1} : n \in \mathbb{N}, n > 2\}$.(no proof needed)

   (b) Let $f : [0, \infty) \to [0, \infty)$ be defined by $f(x) = x^2$. Prove $f$ is bijective (a proof is needed for this one).

3. Let $(X, d)$ and let $(X, e)$ be metric spaces. That is, $d$ and $e$ are metrics on the same set $X$. Define the function $h : X \times X \to \mathbb{R}$ by $h(x, y) = d(x, y) + 2e(x, y)$. Prove $h$ is a metric on $X$.

4. Let $(\mathbb{R}^2, e)$ be the metric space where $e$ is the Euclidean metric and let $(\mathbb{R}, d)$ be the metric space where $d$ is the Euclidean metric. Define

$$S = \{(x, 2 - 3x) \in \mathbb{R}^2 : x \in \mathbb{R}\}.$$

Note $(S, e)$ is a metric space. Show $\phi : S \to \mathbb{R}$ is an isometry where

$$\phi(x, y) = x - 3y.$$

5. Note that $C[0,1]$ is a metric space with metric $d(f,g) = \int_0^1 |f - g| dx$. Define $S = \{f(x) = x(1 - x^n) : n \in \mathbb{N}, n > 2\}$ a subset of $C[0,1]$.

   (a) Compute $d(f,g)$ where $g(x) = 0$ and $f(x) = x(1 - x^n)$ where $n \in \mathbb{N}, n > 2$.

   (b) Compute $\operatorname{dist}(g, S)$ where g and S are defined as above.

   (c) Is $g \in iso(S)$? Is $g \in acc(S)$? Explain.

6. Define $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by the formula

$$d((x_1, y_1), (x_2, y_2)) = (\sqrt{|x_1 - x_2|} + \sqrt{|y_1 - y_2|})^2.$$

(a) Compute $d((1, 3), (1, 2))$ and $d((2.2), (1, 2))$.

(b) Prove $(\mathbb{R}^2, d)$ is a metric space or prove it is not a metric space.

**Take Home:** A common and interesting example of higher mathematics being interpreted for non-mathematicians is the Fixed Point Theorem. There are many versions of this theorem and later in this course we will see some of the mathematics behind this famous theorem. Write an essay to explain the fixed point theorem to your high school students. Some common and fun methods to explain the theorem

- traveling monk
- stirring coffee
- stretching a pizza dough, and
- placing a map of New York onto the ground.

You only need to give a lay person explanation of the theorem. The name of the theorem as you may see it is the Brouwer fixed point theorem and we will learn the Banach fixed point theorem.