Complex Numbers

Mandelbrot Set

Julia set of Quadratic Polynomials

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for math club

This presentation contains parts of the following sources

From $Youtube.com$

Dimension 5, 6

Lecture movies by Adrien Douady

From Prof. Bob Devaney's site Boston University

http://math.bu.edu/DYSYS/FRACGEOM/index.html

For slides, their slides are prettier than mine. So I am borrowing Bob's slide from his website.

XavierBuff&ArnaudCh´eritat Universit´ePaulSabatier(ToulouseIII)

Before the talk starts

1. Part of Dimension Ep. 5 complex numbers I

https://www.youtube.com/watch?v=2kbM96Jr4nk

2. For 2 min from 8.50 min of Dimensions Ep 6

https://www.youtube.com/watch?v=XzT5XSgkLvk

From 8 min 50 second

After the talk

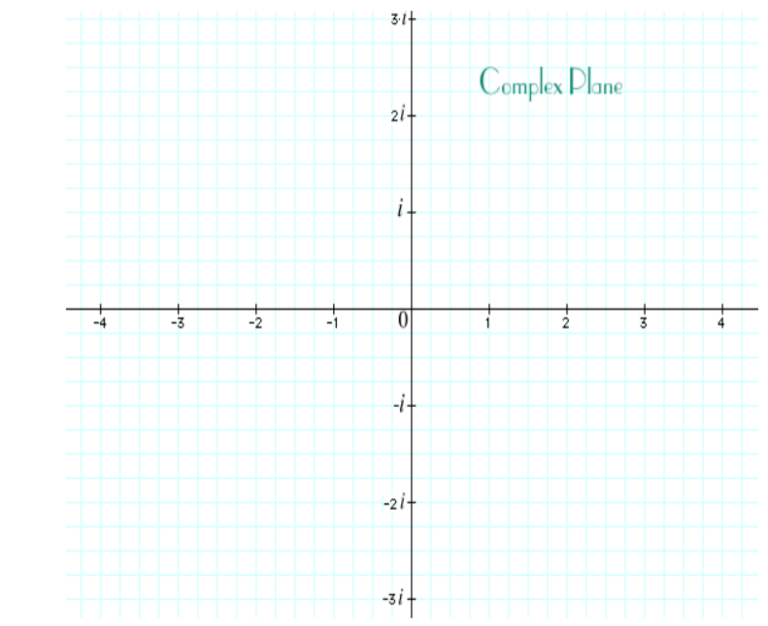
we continue watch Ep. 6 to the end

https://www.youtube.com/watch?v=onMLujxxwug

In the usual real plane R^2,
Add component wise and

Give life of algebra by :

$$i^2 = -1$$



Arithmetic in complex plane C

Consider i as a variable in college algebra

Whenever you see

$$i^2$$

Substitute i^2 by -1.

$$(a + bi) + (c + di)$$
$$= (a + c) + (c + d)i$$

For example,

$$(2 + i) + (3 + 7i)$$
$$= 5 + 8i$$

$$(2 + i) + (3 + 5i)$$

$$= (2 + 3) + 5i$$

$$= 5 + 8i$$

$$i^3, i^4$$

$$i^3 = i^2 \cdot i$$

$$= (-1)i$$

$$= -i$$

$$\mathrm{i}^4 = \left(i^2\right)^2 = (-1)^2 = 1$$

Example (FOIL)

$$(2+i)\cdot\ (3+5i)$$

$$= 2(3+5i) + i(3+5i)$$

$$= (2\cdot 3 + 2\cdot 5i) + (i\cdot 3 + i\cdot 5\cdot i)$$

$$= (6+10i) + (\ 3\cdot i + 5\cdot i\cdot i)$$

$$= (6+10i) + (\ 3\cdot i + 5(-1)$$

$$= (6-5) + (\ 10i + \ 3\cdot i\ )$$

$$= 1 + 13i$$

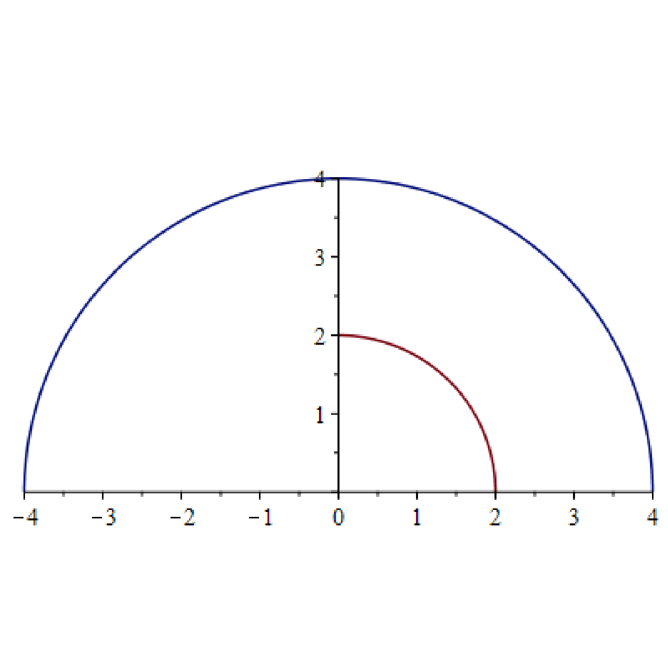
Beauty of Multiplication of two complex numbers

- Radius (length) are multiplied
- Angles are added.
- Richard Feynman once said the most beautiful formula …

Euler's formula:

$$r_1 e^{ia} \cdot r_2 e^{ib} = r_1 r_2 e^{i(a+b)}$$

writing $\cos a + i \sin b$ *as* e^{ia}



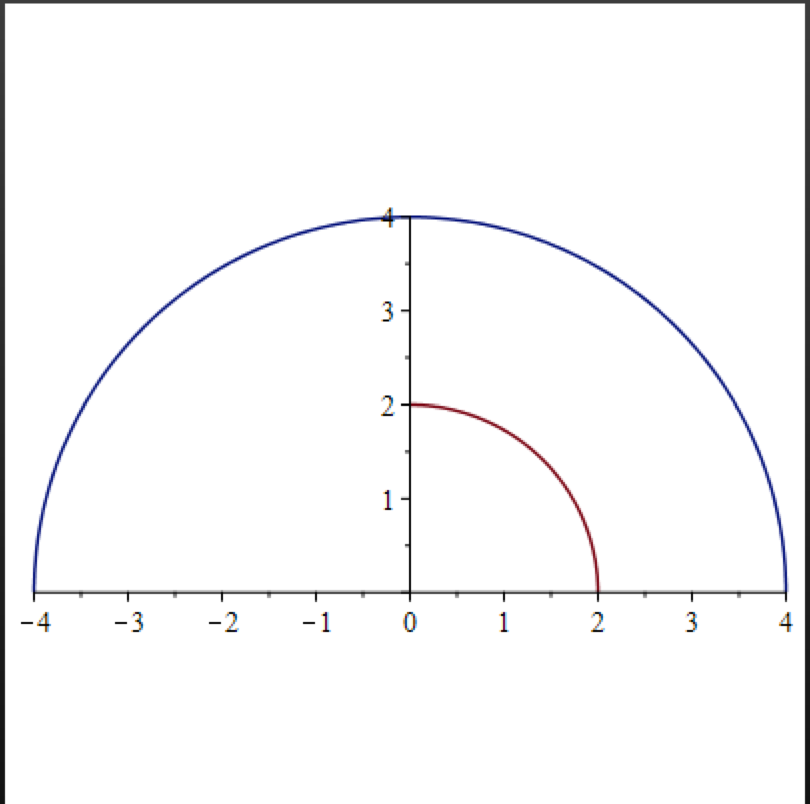
Beauty of Multiplication of two complex numbers

When a complex number

$$z = r((\cos\theta) + i\sin\theta)$$

is squared, Radius is squared and angle is doubled.

Notice that the quarter circle with radius 2 is mapped to the half circle with radius 4.



The algebra

$$i^2 = -1$$

brings

beautiful

and

rich

structure

to mathematics

Iterates of a function: repeated Composition by itself

$$f : C \to C$$

$$(f \circ f)(z) = f\big(f(z)\big)$$

$$(f \circ f \circ f)(z) = f(f \circ f(z)) = f(f(f(z)))$$

Write iterates using power

$$f^{(1)}(z) = f(z)$$

$$f^{(n)}(z) = f\left(f^{(n-1)}(z)\right)$$

$$= f\left(f(f \cdots (z))\right)$$

$$f : C \to C$$
$$f(z) = z^2 - 2$$

$$f^{(1)}(0) = f(0) = -2$$

$$f^{(2)}(0) = f\left(f^{(1)}(0)\right) = f(-2)$$
$$= (-2)^2 - 2 = 2$$

$$f^{(3)}(0) = f(2) = 2^2 - 2 = 2$$

$$f^{(4)}(0), = 2 \ \cdots,$$

Iterates of $f^{(n)}(0)$, $f(z) = z^2 - 2$			
$f^{(1)}(0)$	$f(0)$	$0^2 - 2$	-2
$f^{(2)}(0)$	$f\left(f^{(1)}(0)\right)$	$f(-2)$	2
$f^{(3)}(0)$	$f\left(f^{(2)}(0)\right)$	$f(2)$	2
$f^{(4)}(0)$		$f(2)$	2
$\vdots$		$\vdots$	$\vdots$

$Definition:$ fixed point

z is a fixed point of $f(z)$ iff

z=$\mathrm{f}(\mathrm{z})$

$z = 2$ *is a fixed point of* $f(z) = z^2 - 2$

$\mathbf{0}$ and $-\mathbf{2}$ are preimages of a fixed point of $z^2 - 2$.

More example: $f(z) = z^2 + i$

$$f^{(1)}(0) = f(0) = 0^2 + i = i$$

$$f^2(0) = f(f(0)) = i^2 + i = -1 + i$$

$$f^3(0) = f\left(f^{(2)}(0)\right) = f(-1 + i)$$

$$= (-1 + i)^2 + i$$

$$= ((-1)^2 - 2i + (i)^2 + i$$

$$= 1 - 2i - 1 + i = -i$$

$$f(z) = z^2 + i$$

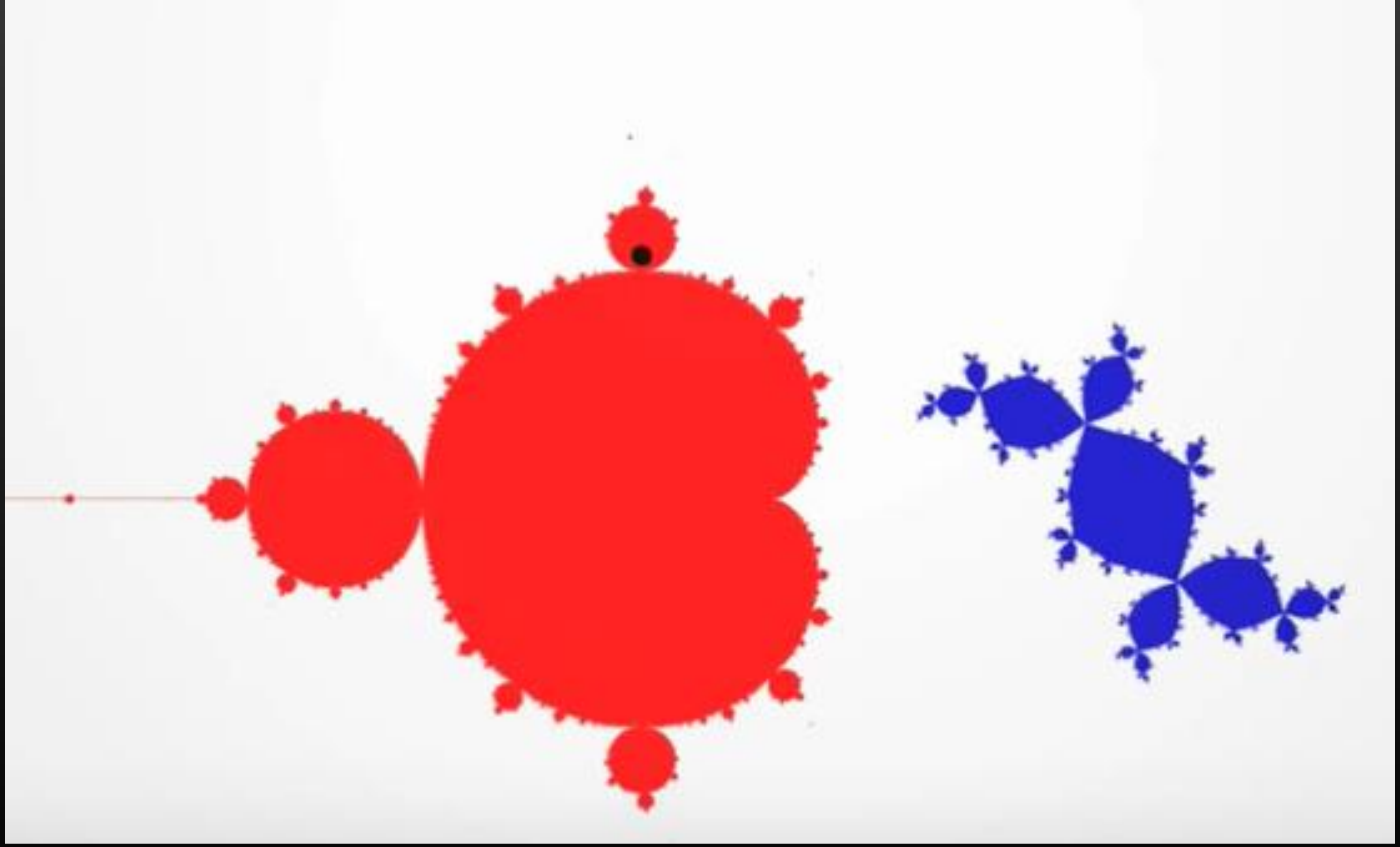
Every two periods -1+i and i are repeated!!!

- $f(0) = i$
- $f^{(2)}(0) = -1 + i$
- $f^{(3)}(0) = -i$
- $f^{(4)}(0) = -1 + i$
- $f^{(5)}(0) = -i$
- $f^{(6)}(0) = -1 + i$
- $f^{(7)}(0) = -i$
- …
- $-1 + i$ and $-i$ are called periodic points with period 2 of the polynomial $f(z) = z^2 + i$
- i is preperiodic: it is a preimage of a periodic point.

Definition: Periodic point , pre-periodic point

- z is a periodic point of f if $f^{(k)}(z) = z$
- **The smallest positive number k is called the period of z**
- **Preimages of periodic point is called pre-periodic point.**
- **For $f(z) = z^2 + i$**
  - $-1 + i$ **and** $-i$ **are periodic points with period 2**
  - i **is preperiodic**

In the movie, the red set is called Mandelbrot set and the red set is called the filled in Julia set of $z^2 + c$, where c is the point where the black dot in the red set is pointing.



Definition

Mandelbrot set M

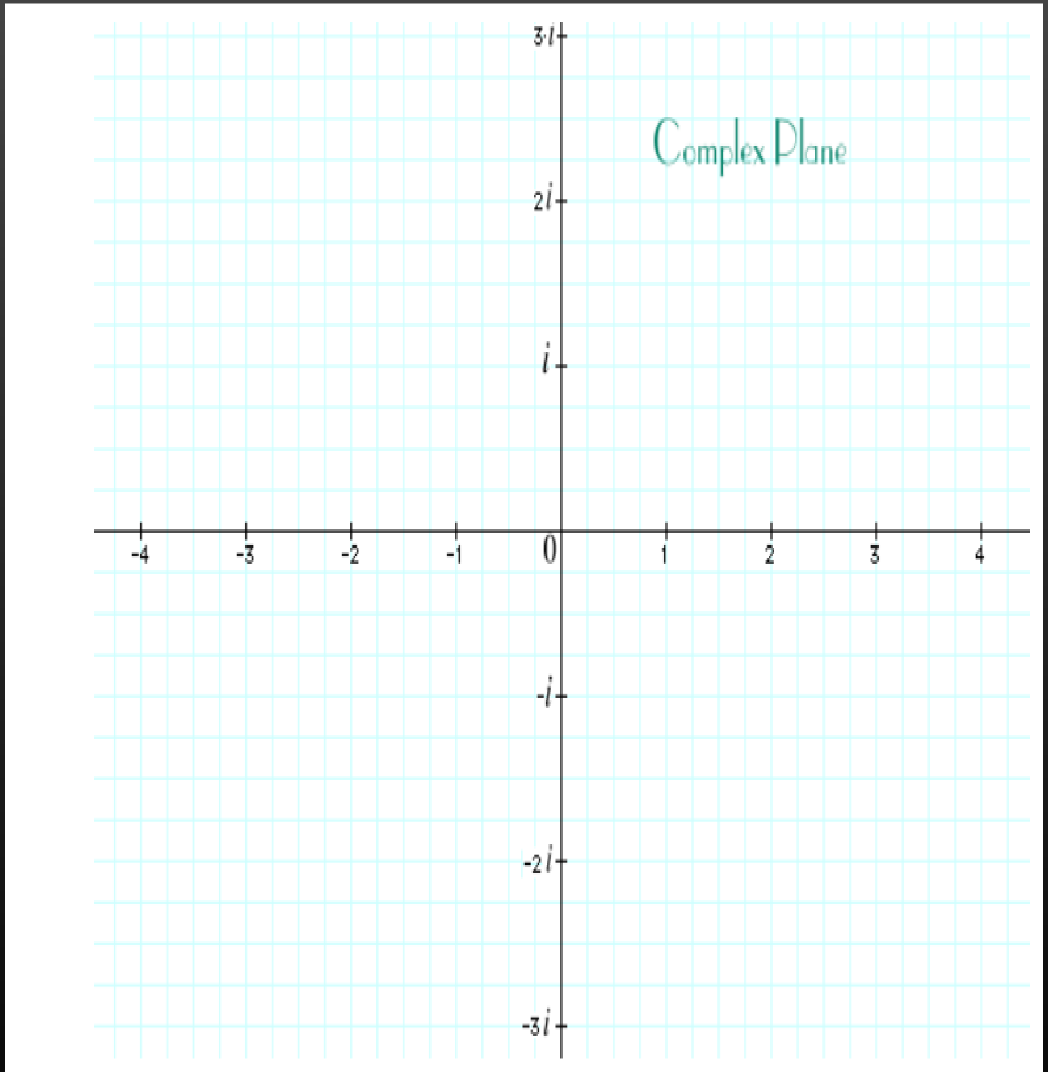
Denote

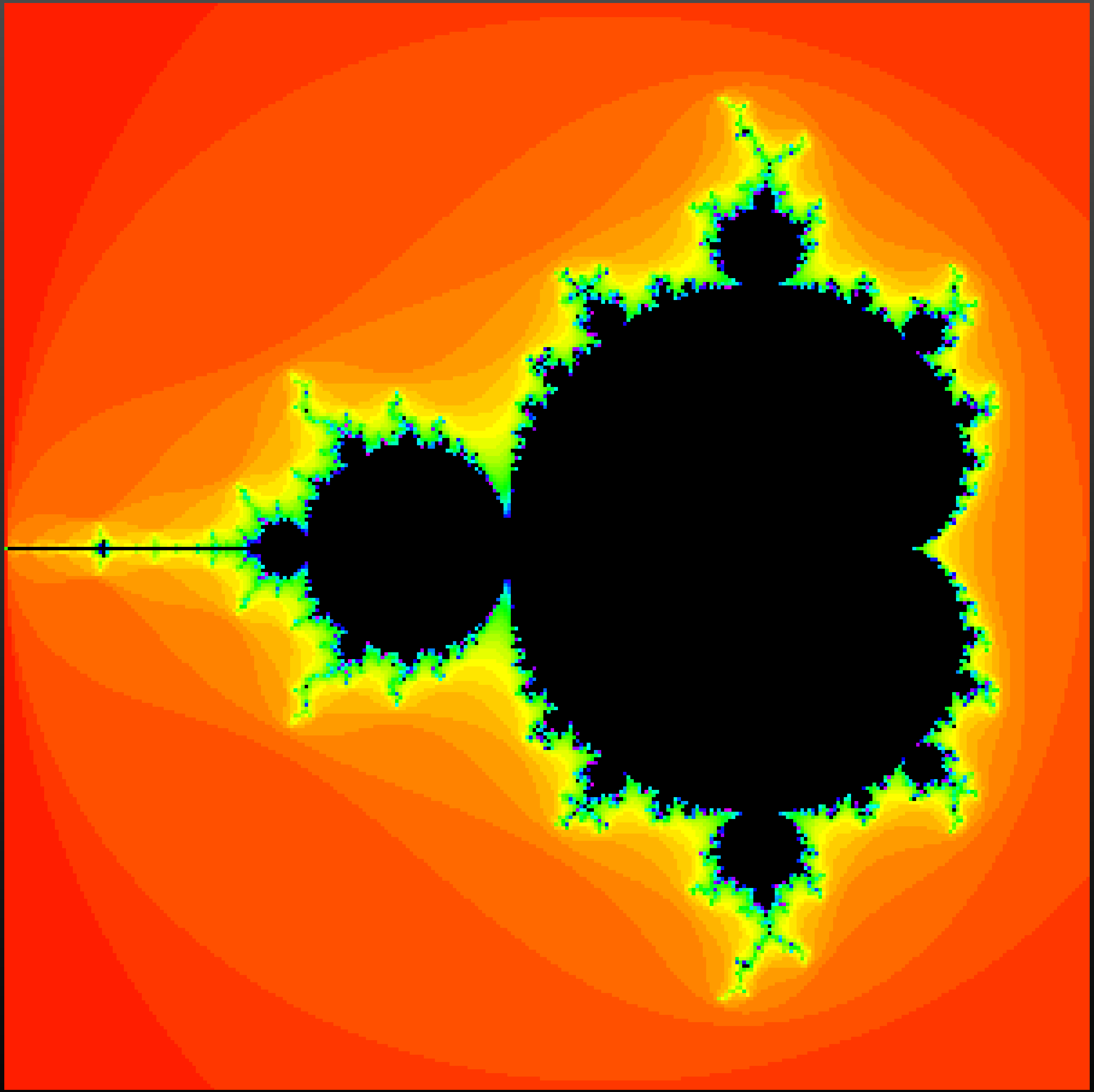
$$f_c^{(n)} = z^2 + c$$

$$M = \{c \in \mathbb{C} \mid |f_c^{(n)}(0)| \text{ is bounded}\}$$

It is known that

$$c \in M \; iff \; \limsup_{n\to\infty} |f_c^{(n)}(0)| \leq 2$$


Complex Plane
3i
2i
i
-i
-2i
-3i
-4
-3
-2
-1
0
1
2
3
4



How to check if it is in M?

- Pick a point c
- $f(z) = z^2 + c$
- Compute $z_n = f^{(n)}(0), n = 1,2,3, \cdots$
- If z_n does not diverges to infinity then it is in M.

How to make Mandelbrot set? How to check if it is in M?

c=0 is in M

$z_0 = 0$

$z_1 = z_0^2 + 0 = 0$

$z_2 = z_1^2 + 0$

$= 0^2 + 0 = 0$

$\dots$

$z_n = 0$

And $|z_n| = 0$ for all n

So c=0 is in M

c=4 is not in M

$z_0 = 0$

$z_1 = z_0^2 + 4 = 4$

$z_2 = 4^2 + 4 = 20 > 4^2$

$z_{n+1} = z_n^2 + c > 4^n$

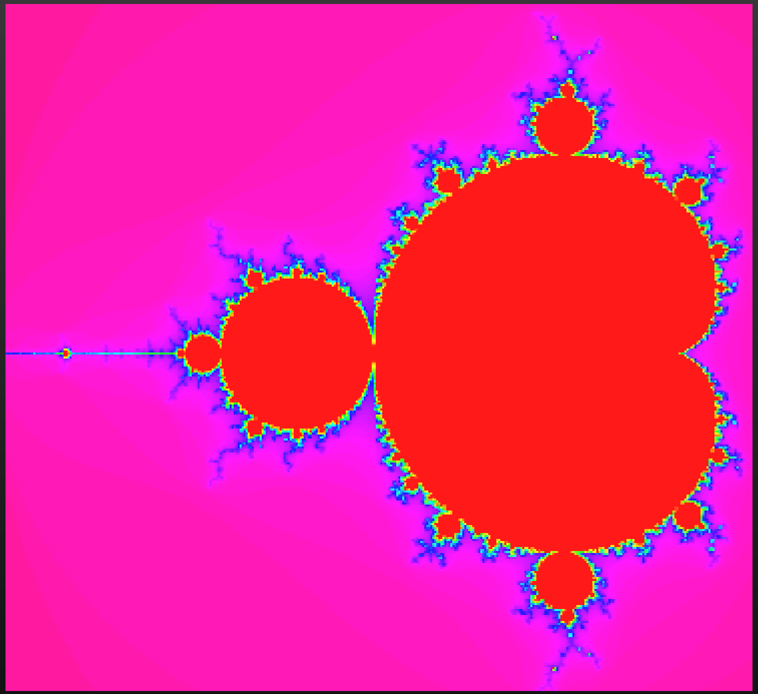
Conclude

$\{|z_n|\}$ is not bounded

So, c=4 is not in M

How does a computer plot Mandelbrot set

```
mandelbrotC := proc(X,Y)
  local Z, ct;
  Z := X + I*Y;
  for ct from 1 while ct<120 and evalf(abs(Z))<2.0
    do
      Z := Z^2 + (X + I*Y)
    od;
  -ct*3;
end:
densityplot('mandelbrotC'(x,y),
  x=-2..0.55, y=-1.15..1.15,
  grid=[500,500],
  scaling=constrained,colorstyle = HUE,
  style=PATCHNOGRID, axes=NONE);
```



How does a computer plot Mandelbrot set

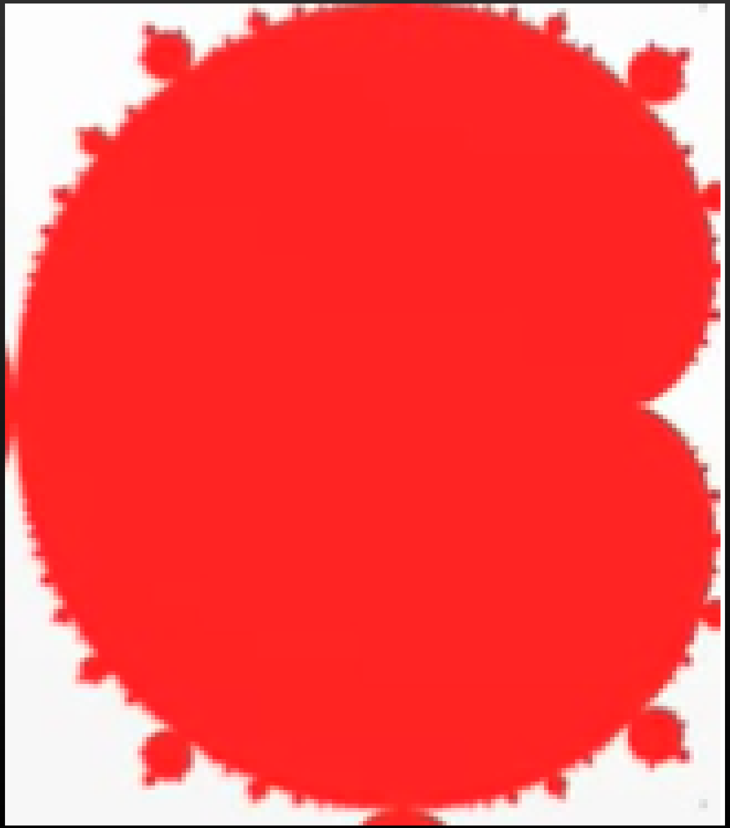
- Density plot in the range: [-2, 0.55]x[-1,5, 1.5]
- Choose a number of grid points, say 200x200.
- Use every grid point as constant value c of f
- Set $f(z) = z^2 + c$
- Compute $z_n = f^{(n)}(0), n = 1,2,3, \cdots, 100$
- If $|z_n| < 2$ plot the grid point c in M.
- Else go to the next grid.

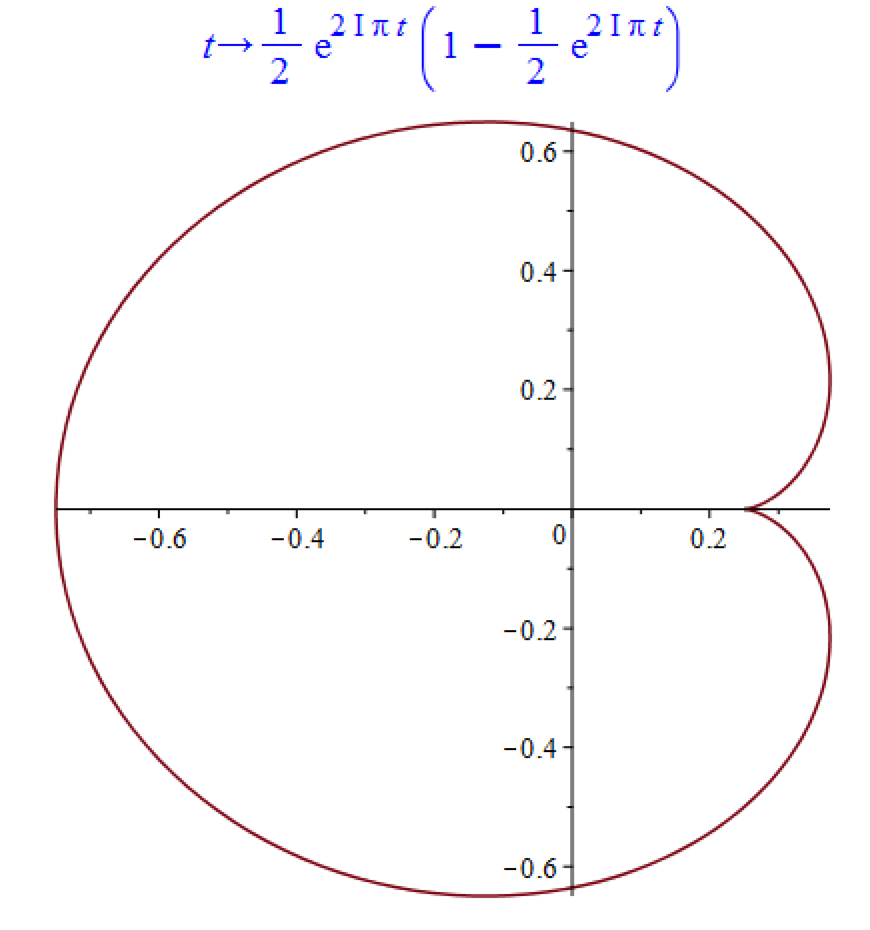
Mandelbrot set

Main Body of M

Cardioid is the locust of

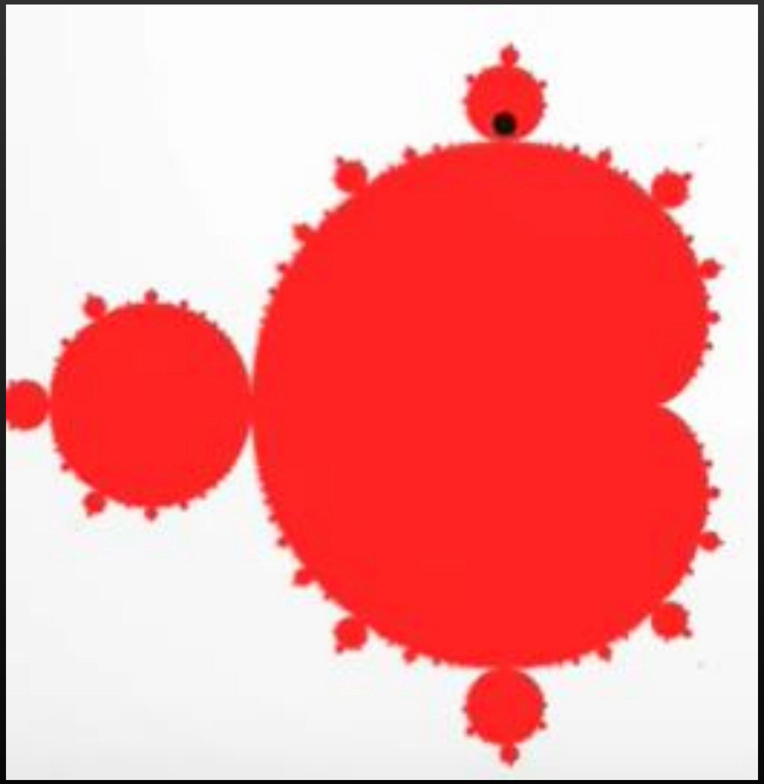
$$C(t) = \frac{e^{2\pi i t}}{2}\left(1 - \frac{e^{2\pi i t}}{2}\right)$$

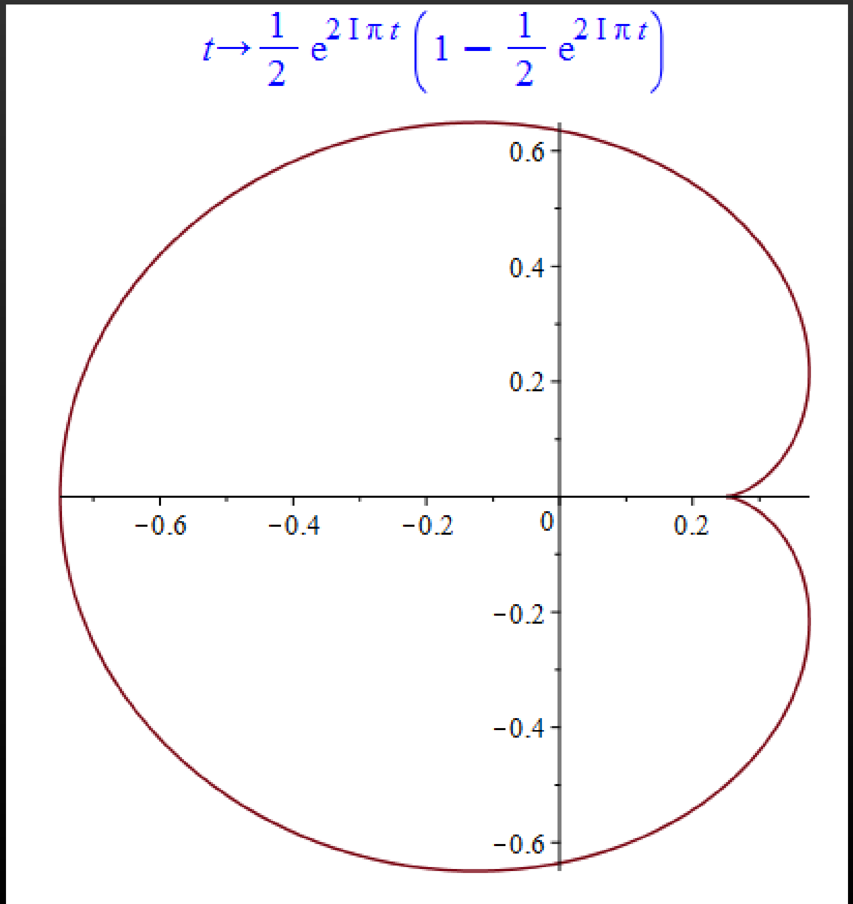




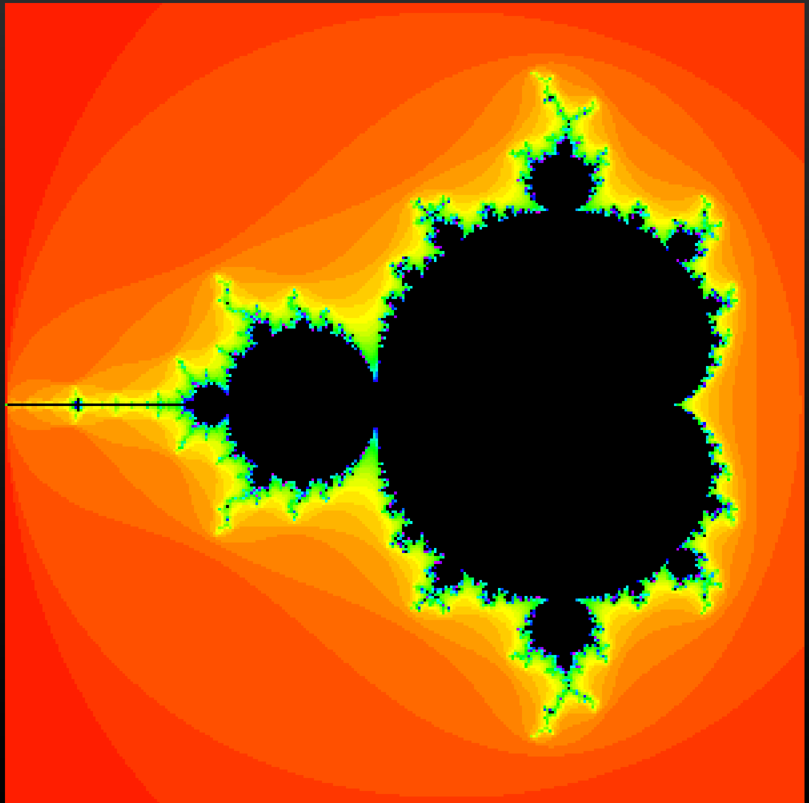
Bulbs are attached at points, $C\left(\frac{p}{q}\right) = \frac{e^{2\pi\frac{p}{q}i}}{2}\left(1 - \frac{e^{2\pi\frac{p}{q}i}}{2}\right)$

Notice that three are rational number many bulb points on the main cardioid





$$c_{\frac{p}{q}} = \frac{e^{2\pi\frac{p}{q}i}}{2}\left(1 - \frac{e^{2\pi\frac{p}{q}i}}{2}\right)$$



$$Crazy\ but\ Interesting$$

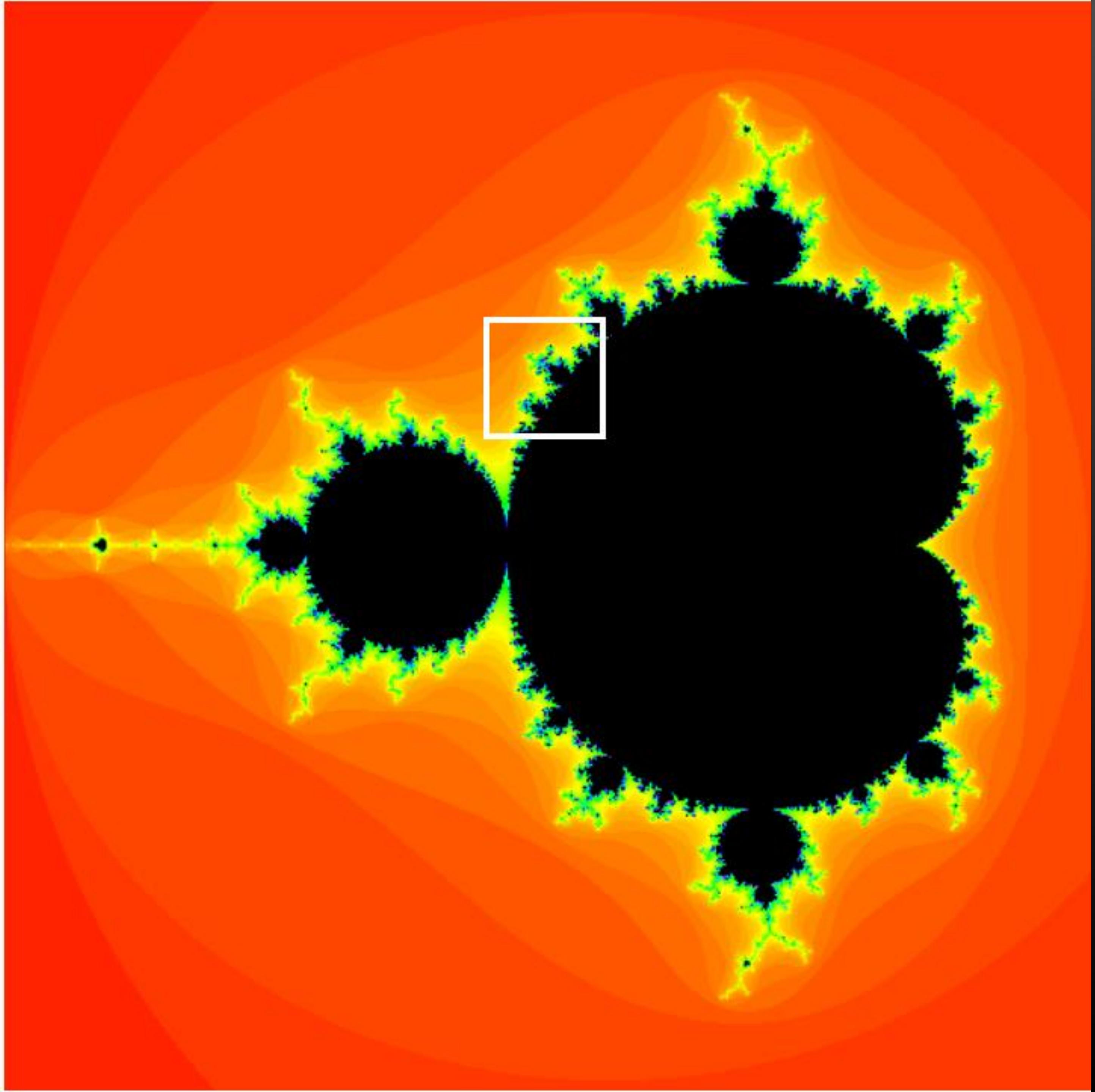
$$Self-Similar$$

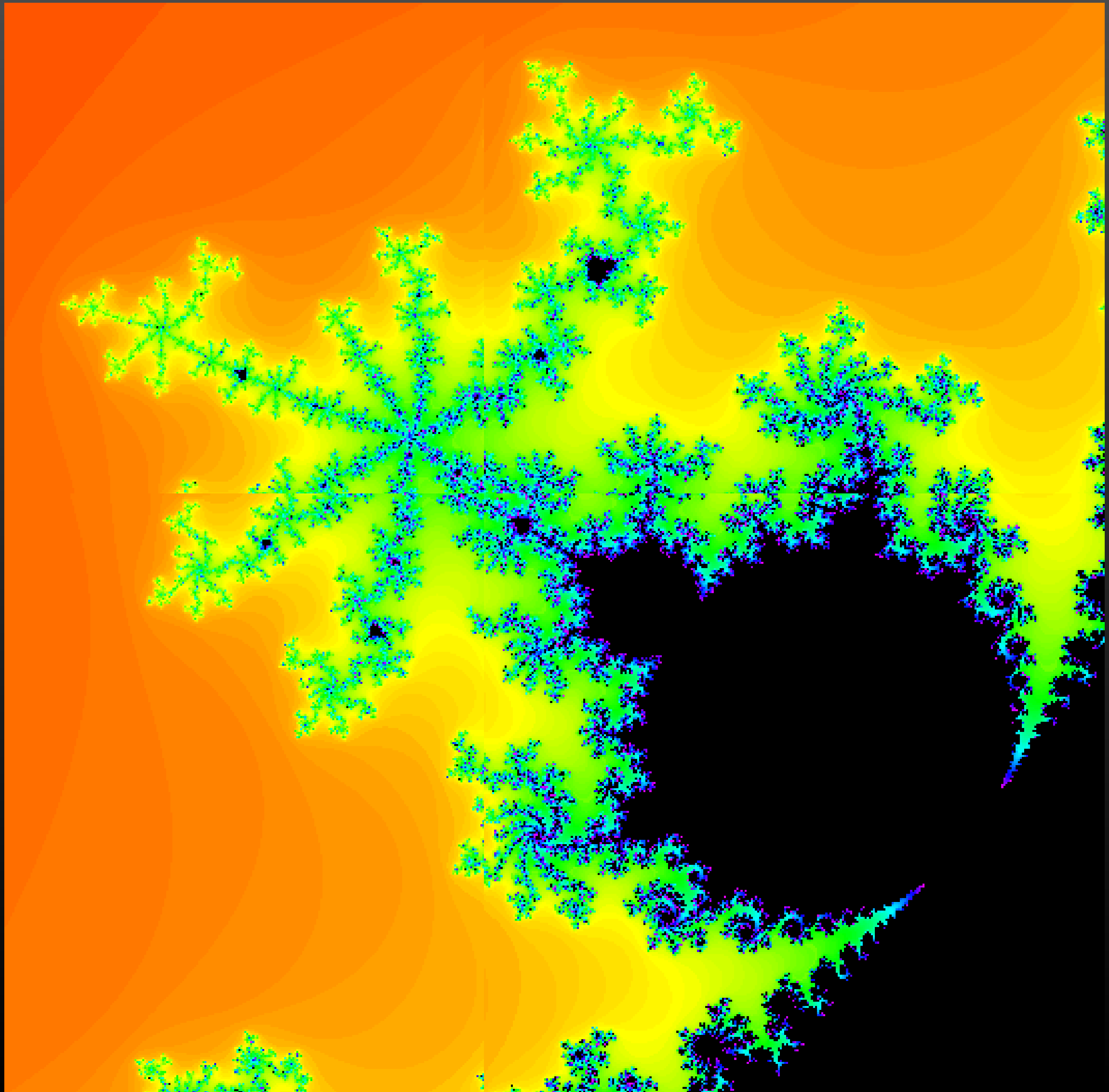
M is Connected

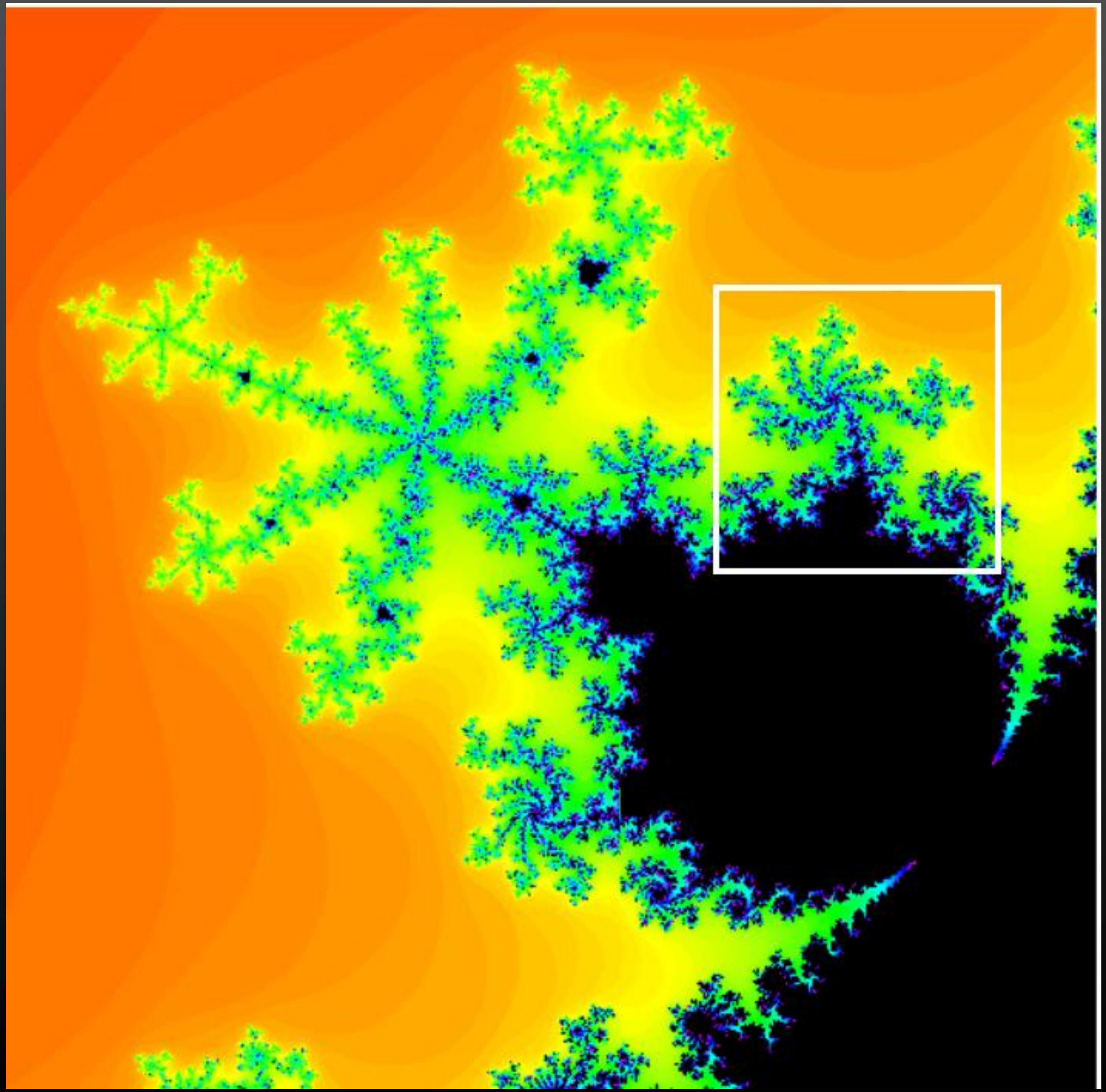
We don't know if M is
locally connectedness.
is still open

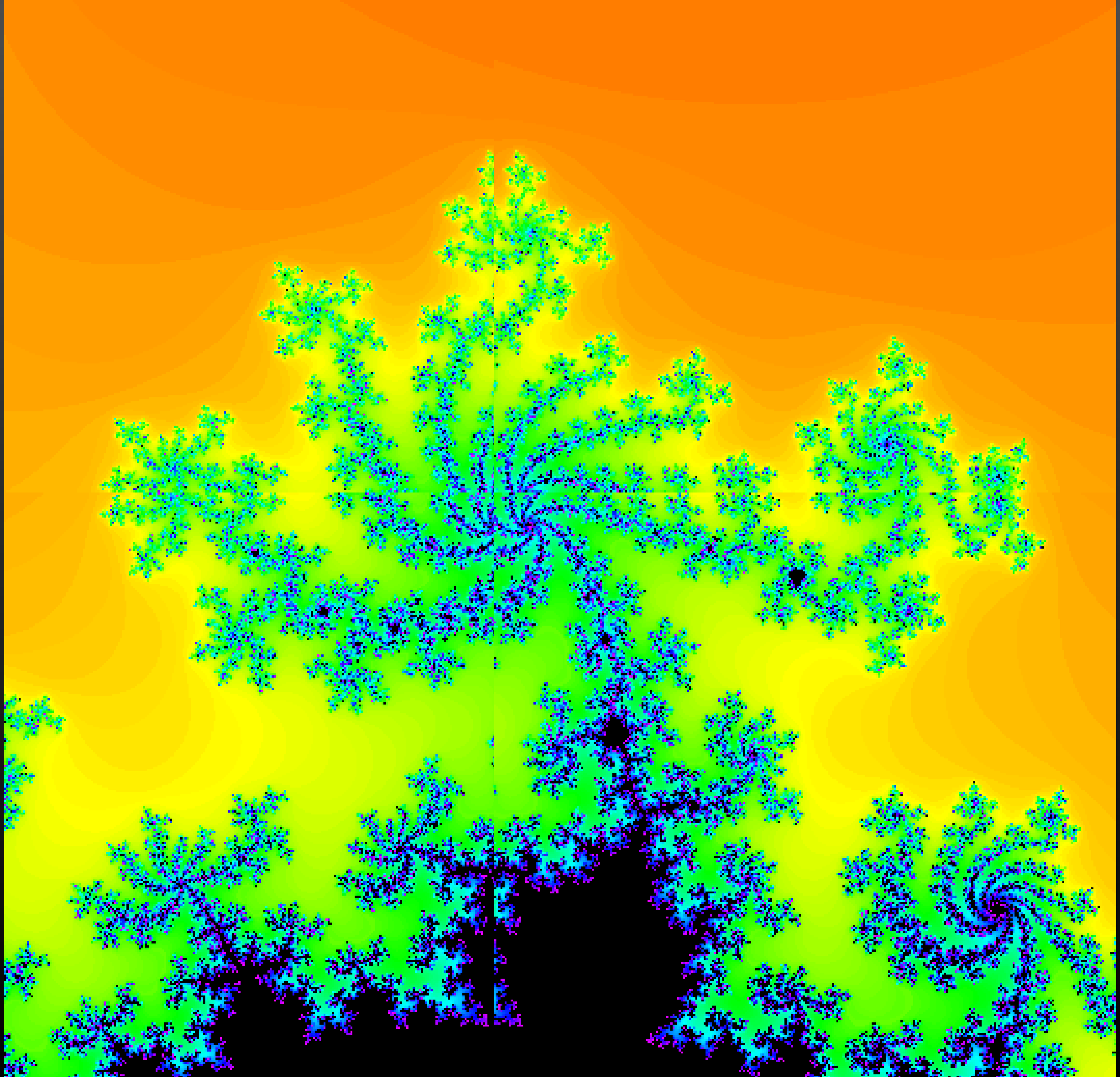
In the sequel, Zoom zoom

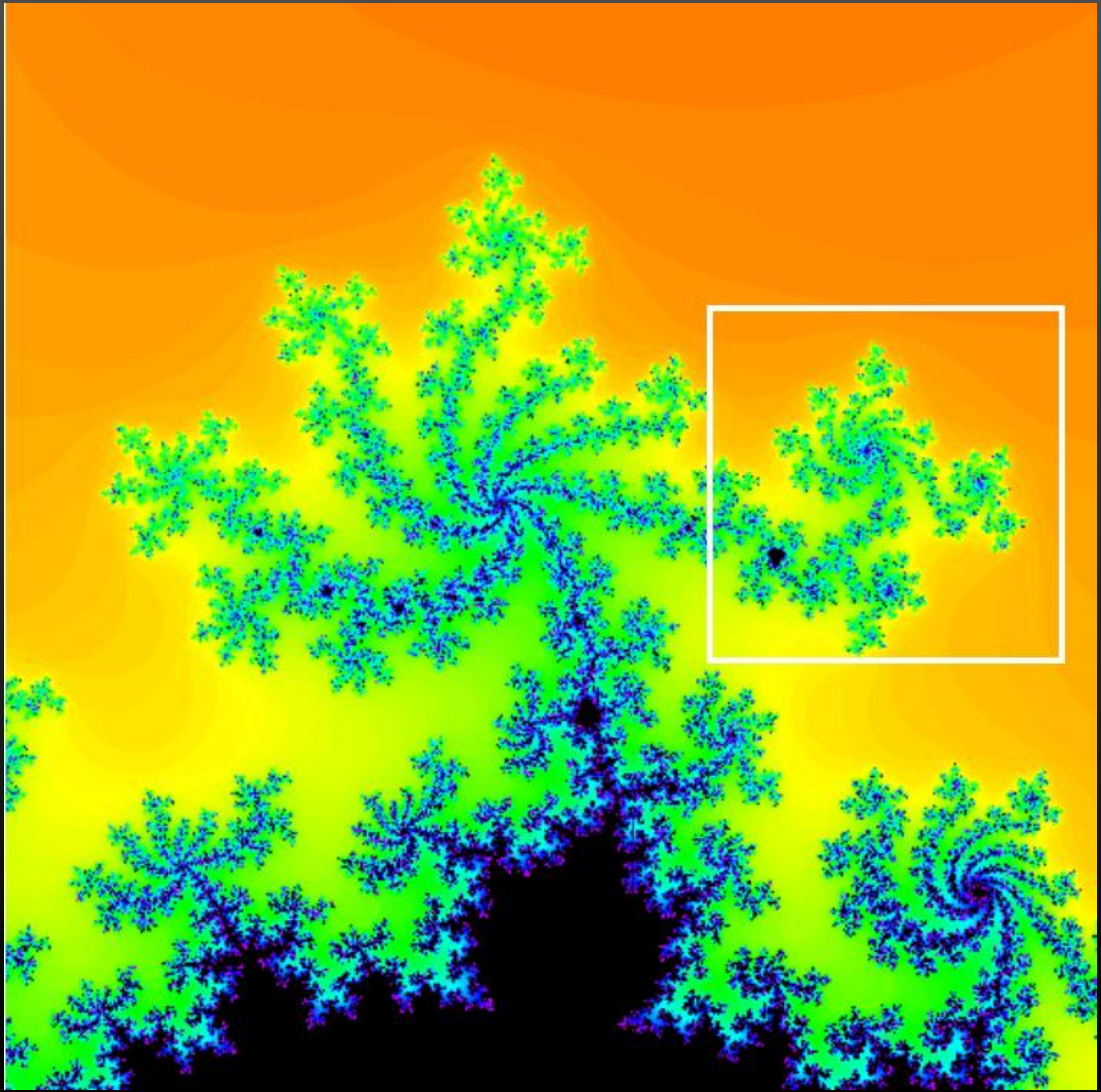
- We will zoom where drawn by squares

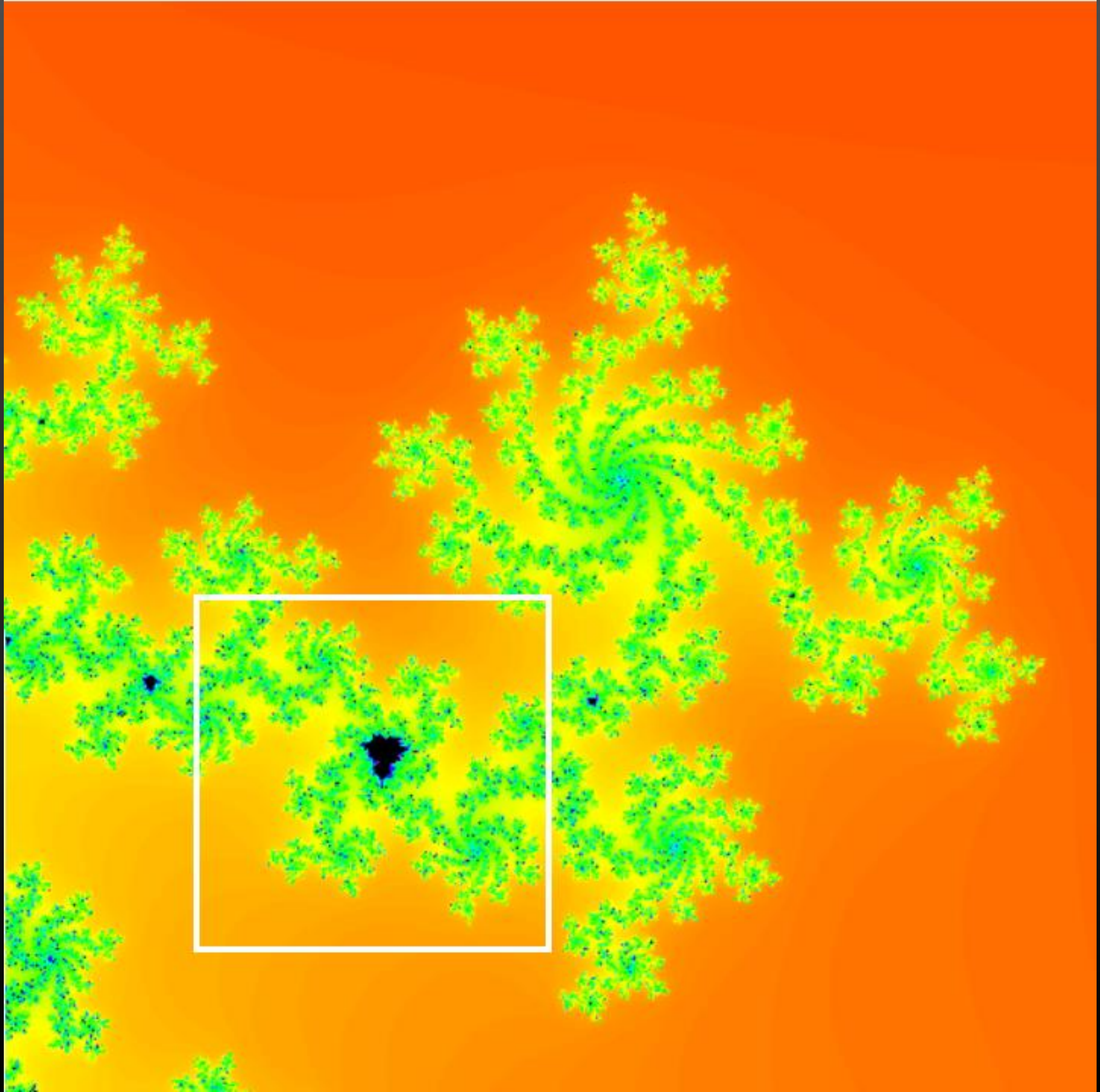


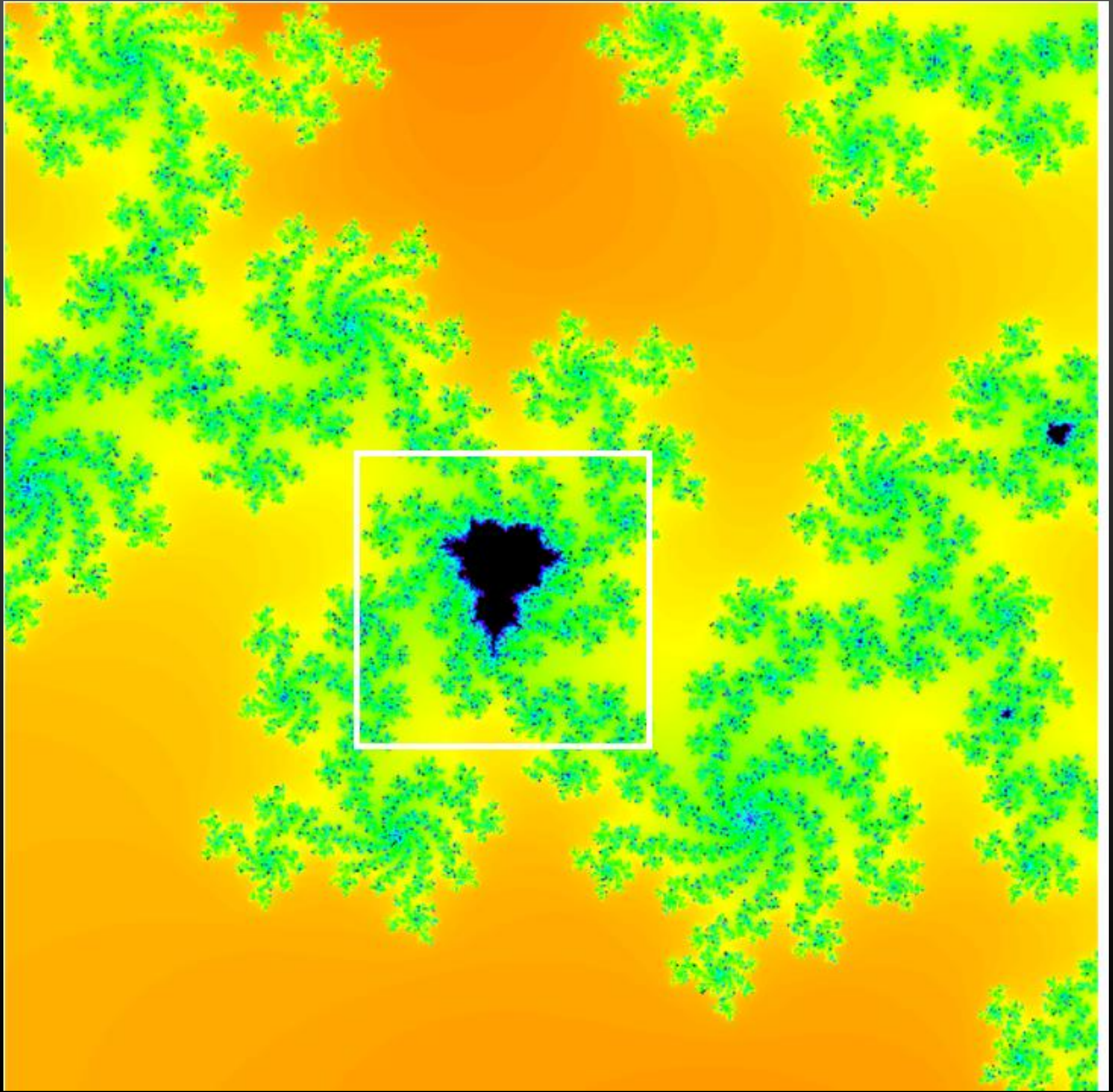


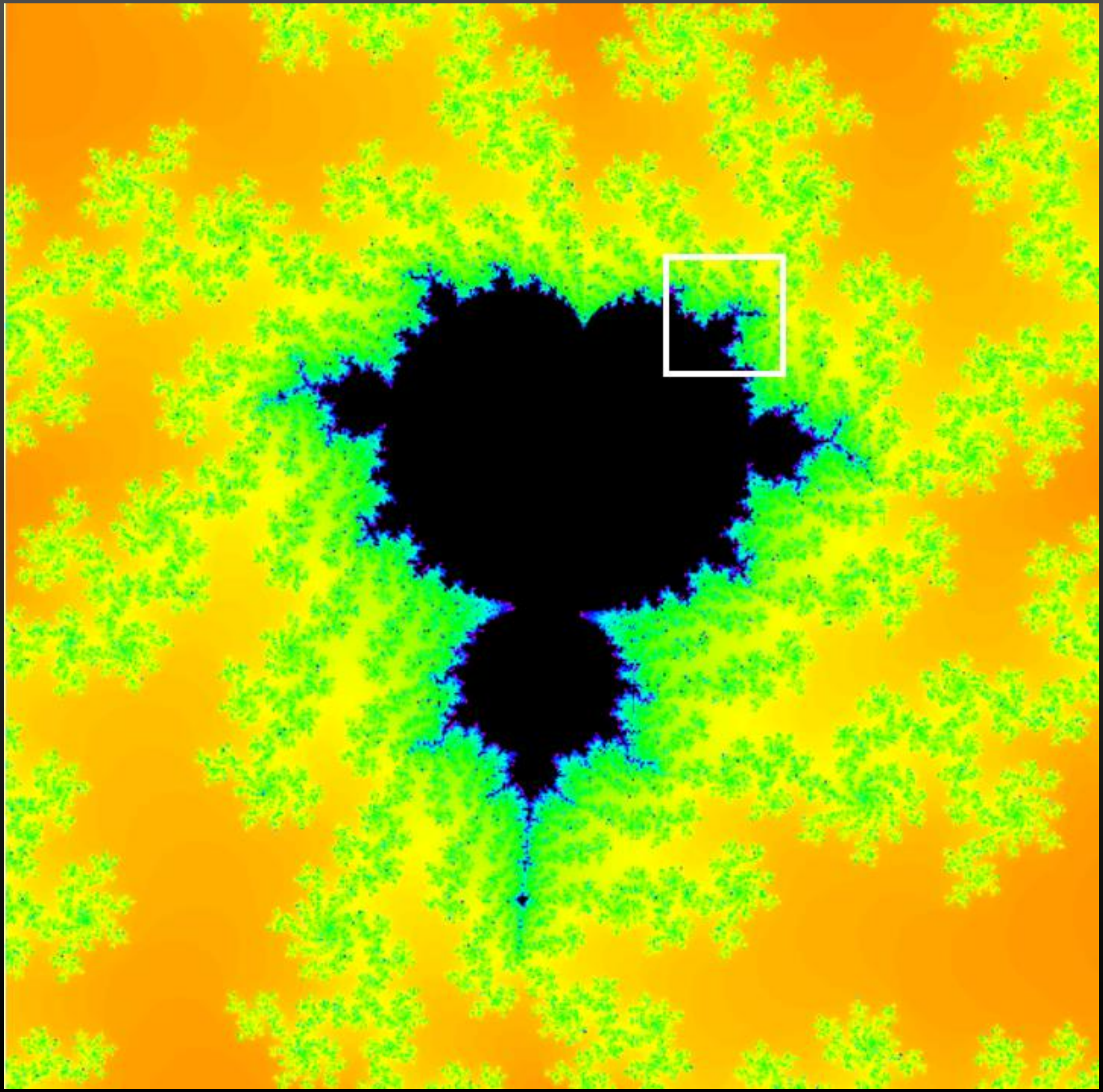


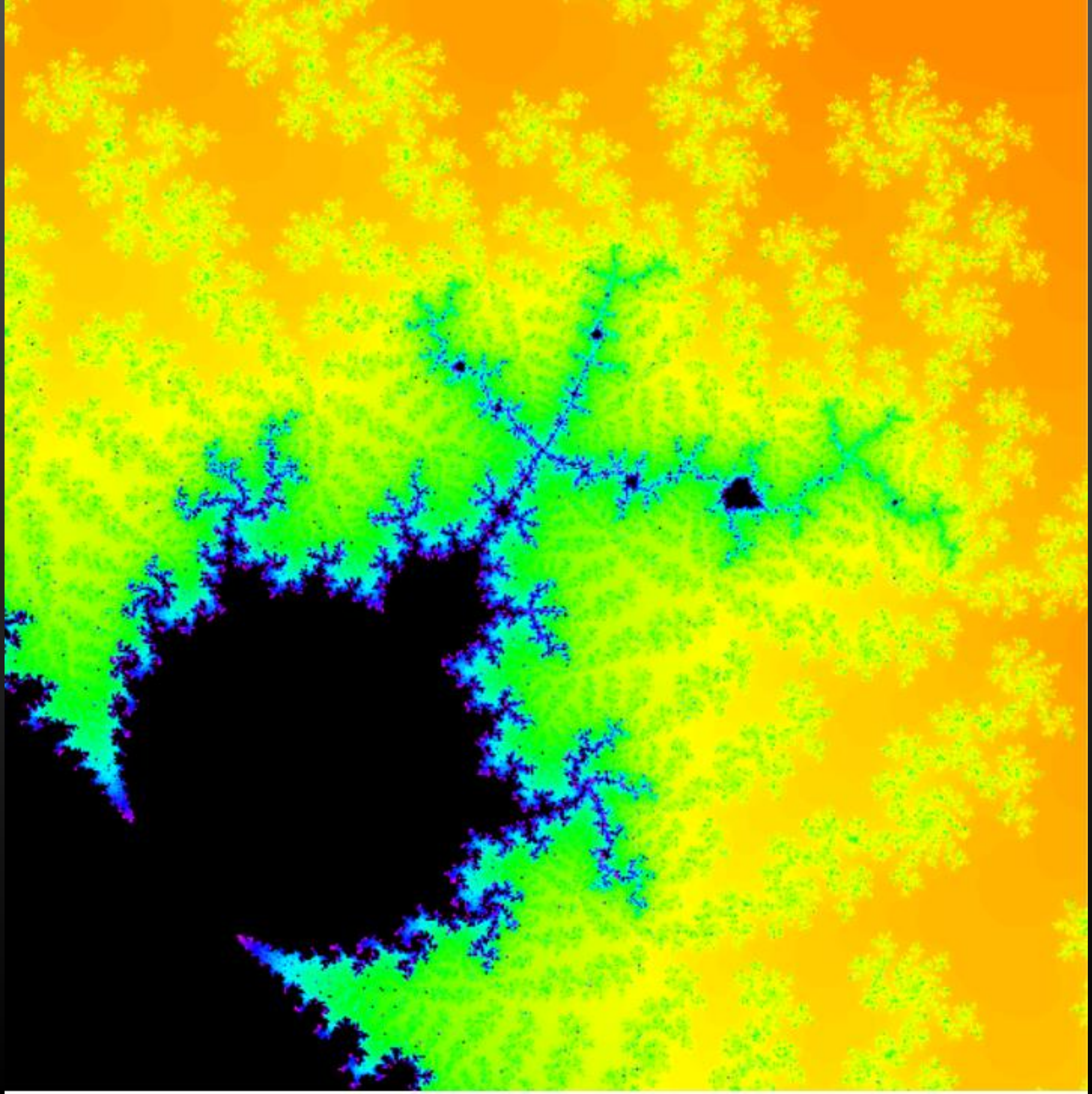




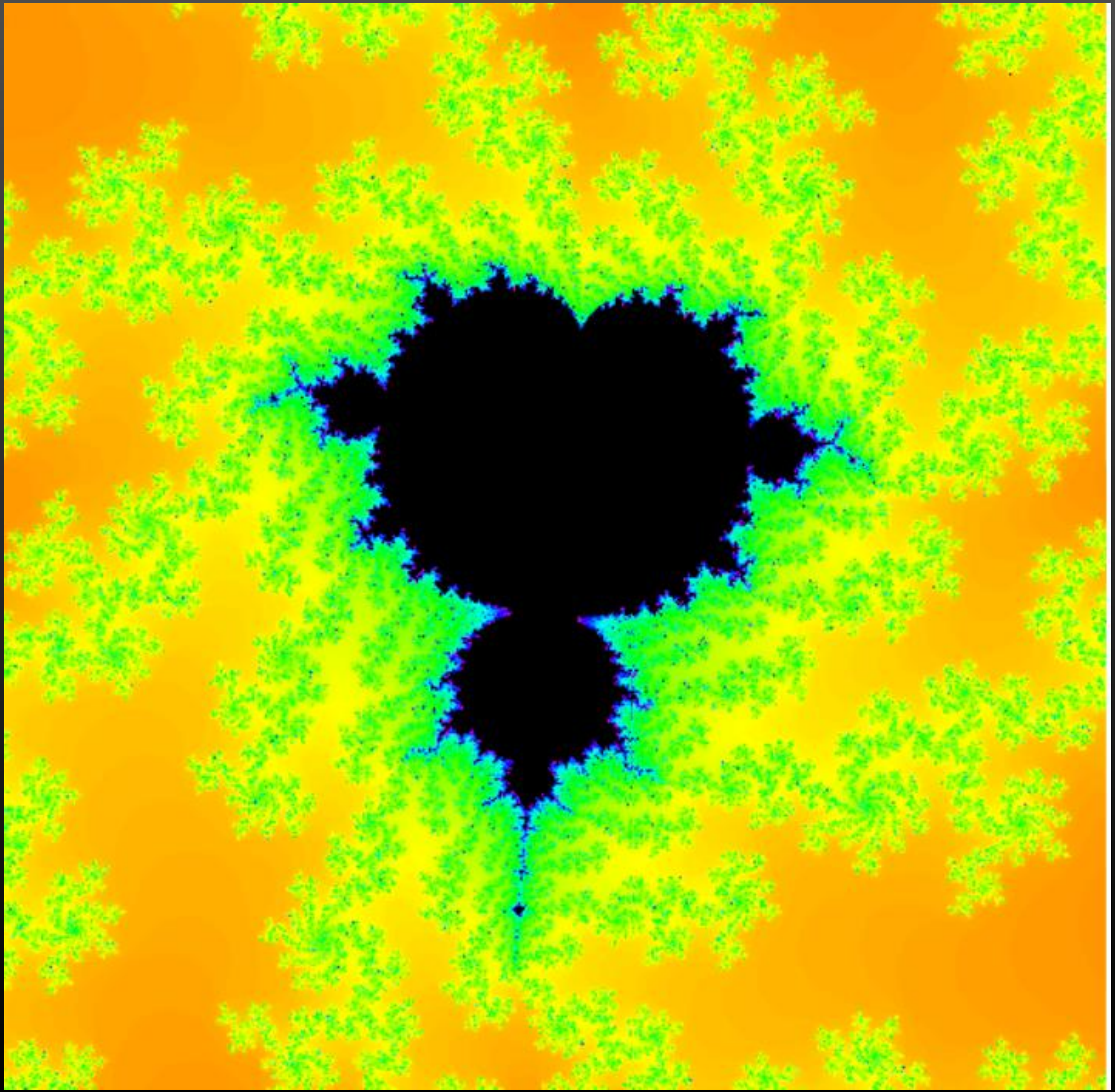




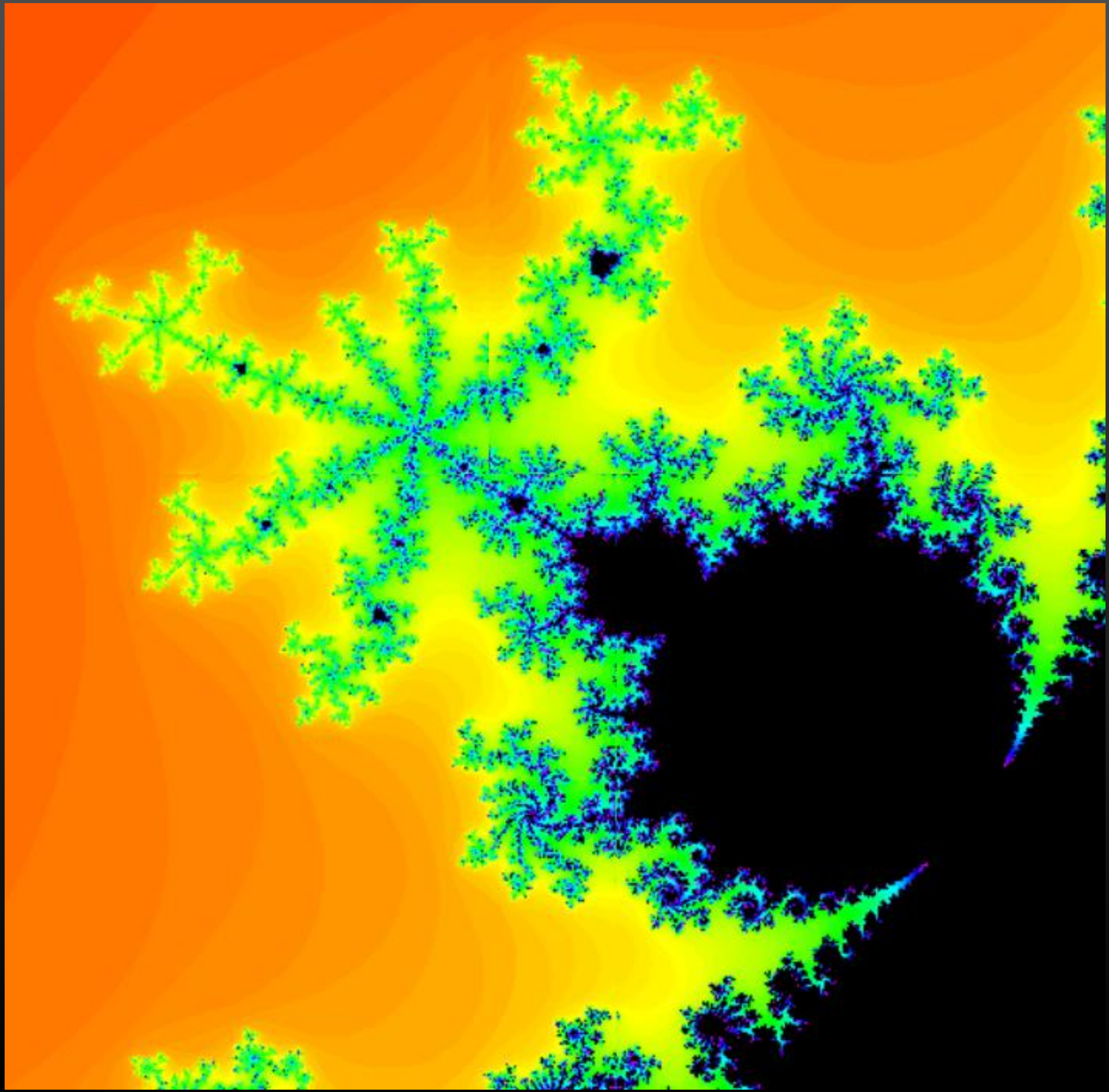




$\frac{3}{4}$ bulb



$\frac{1}{16}$
antenna



$\frac{3}{7}$ bulb

Julia sets

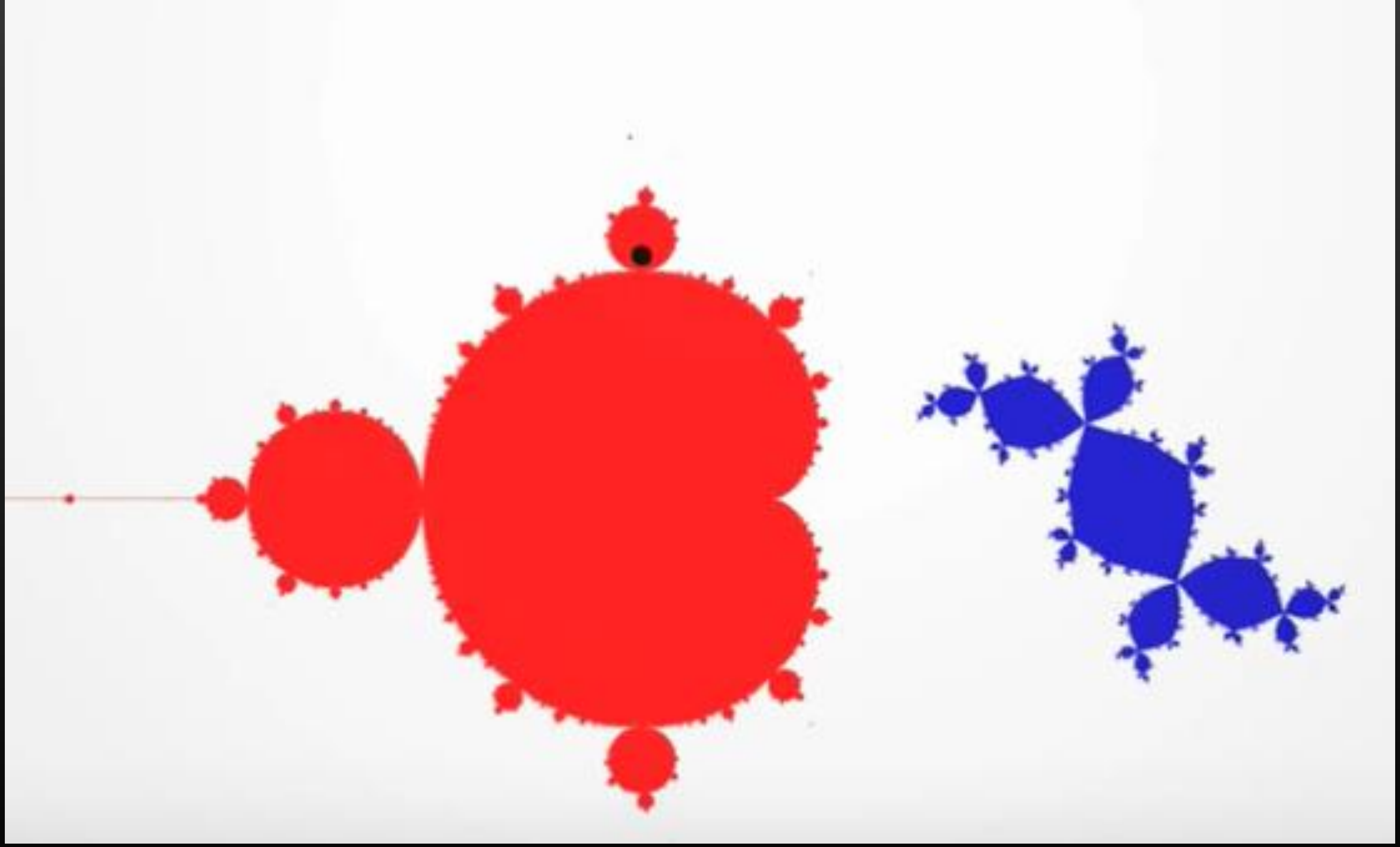
How it looks?

Relation between M and J and FJ

Definitions of filled Julia set and Julia set

What is a Julia set and how to plot them?

$z^2 + c$, where c is the point where the black dot in the red set is pointing



z^2+c
Definitions

$$\boldsymbol{Filled\ Julia} = \{\mathbf{z} \in \boldsymbol{C} |\ \boldsymbol{f}^{(\boldsymbol{n})}(\mathbf{0})\ \nrightarrow \infty\}$$

Julia set is the boundary of Filled Julia Set

Julia set is the closure of repelling periodic points

Fix a complex number c

consider a quadratic polynomial

$f(z) = z^2 + c$

For every $z \in C$,

Compute iterates of $f(z)$

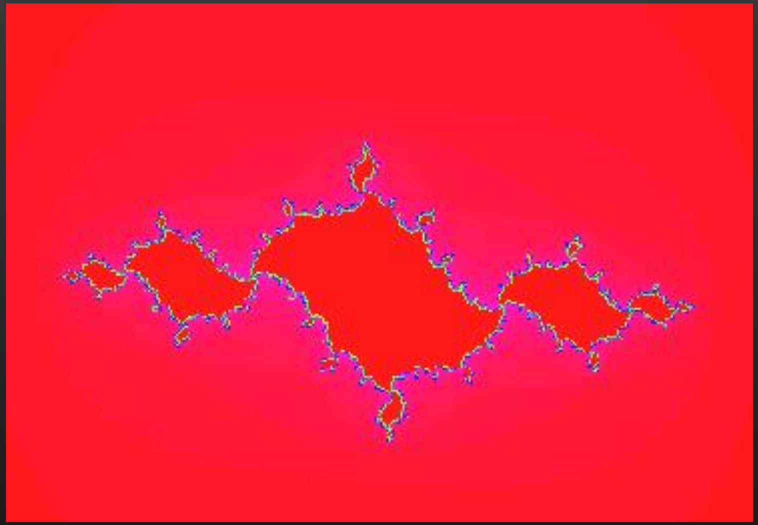
Check

$f^{(n)}(z) \rightarrow \infty$

Maple

$$c = -1.037 + 0.17i$$
$$f(z) = z^2\text{-}1.037 + 0.17*I$$

```
with(plots) :
JuliaSet := proc(X, Y)
 local Z, ct;
 Z := X + I* Y;
 for ct from 1 while ct < 120 and evalf(abs(Z)) < 2.0
  do
   Z := Z^2 -1.037 + 0.17 * I
  od;
 -ct;
end:
densityplot('JuliaSet'(x, y),
 x=-2 ..2, y=-1.5 ..1.5,
 grid= [500, 500],
 scaling= constrained, colorstyle= HUE,
 style= PATCHNOGRID, axes= NONE);
```



Fix a complex number c

consider a quadratic polynomial

$$f(z) = z^2 + c$$

For every $z \in C$,

Compute iterates of $f(z)$

Check

$$f^{(n)}(z) \to \infty$$

Most Julia set we see has Lebesgue measure (area) 0.

Picture we see are filled Julia set
whose boundary is the Julia set.
To visualize Julia set we need to plot filled Julia set.

It is open question if there is a Julia set of quadratic
with positive Lebesgue measure.

Fractal dimension (Hausdorff dimension) is invented

Relation between Julia set of $z^2 + c$ and where in Mandelbrot set

Main Body of M

Cardioid

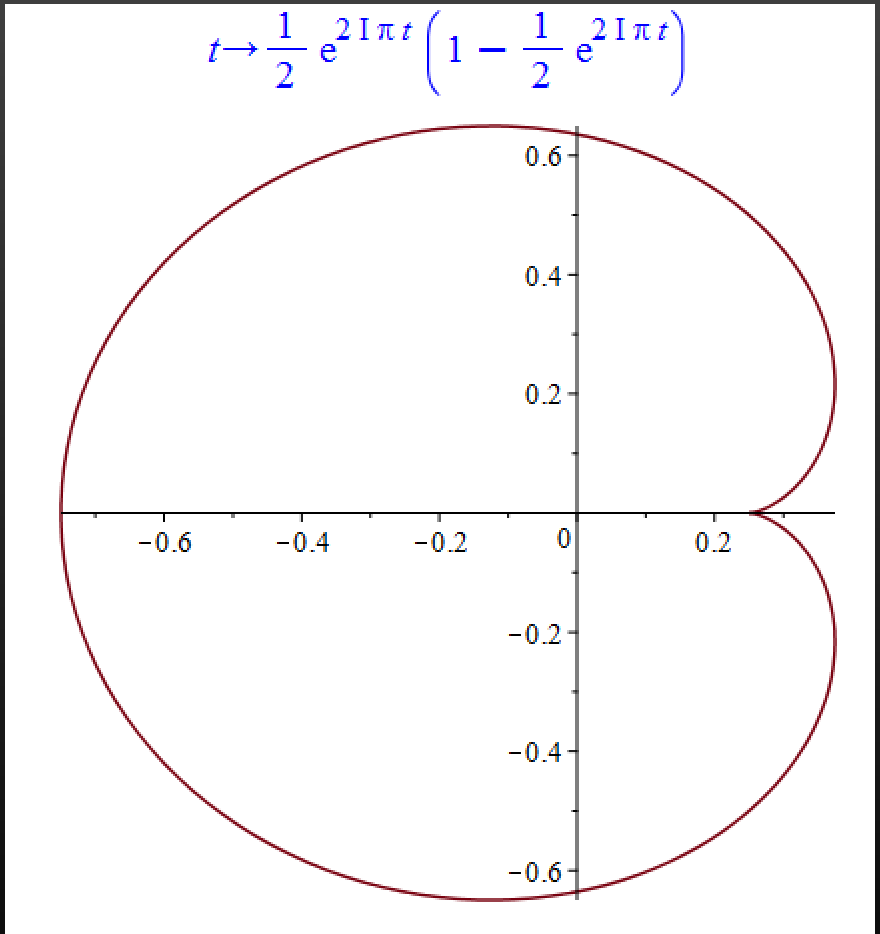
Bulbs are attached at points

$C_{\frac{p}{q}}$ are interesting.

periodic points of $z^2 + c$

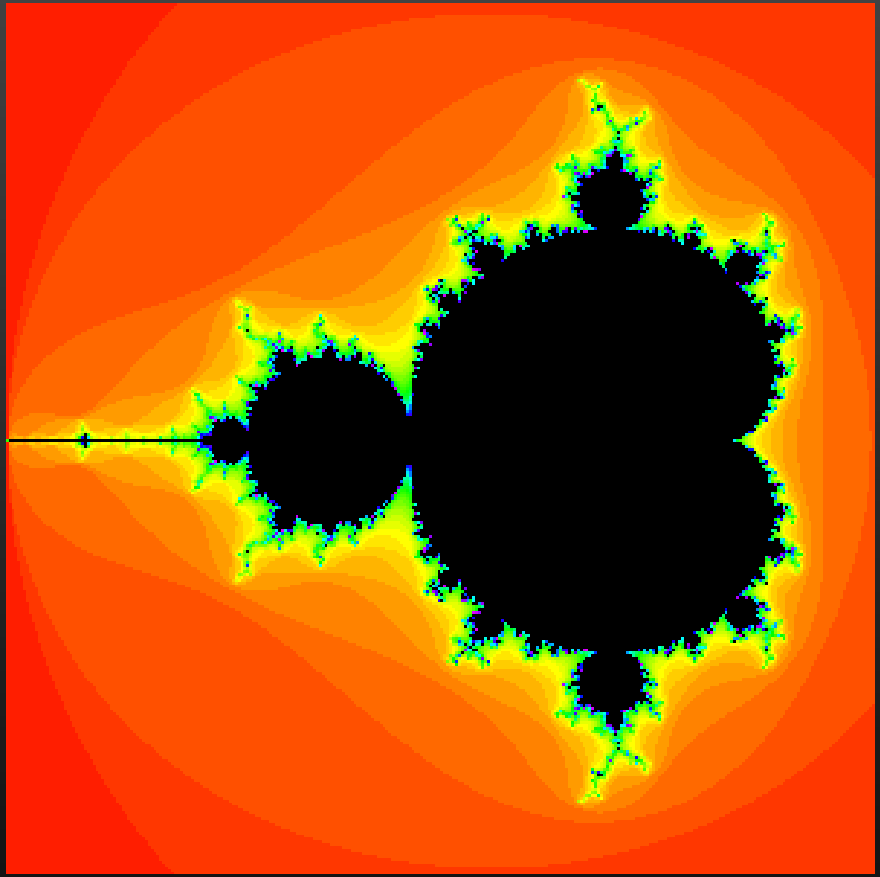
with period q

$$C_{\frac{p}{q}} = \frac{e^{2\pi\frac{p}{q}i}}{2}\left(1 - \frac{e^{2\pi\frac{p}{q}i}}{2}\right)$$

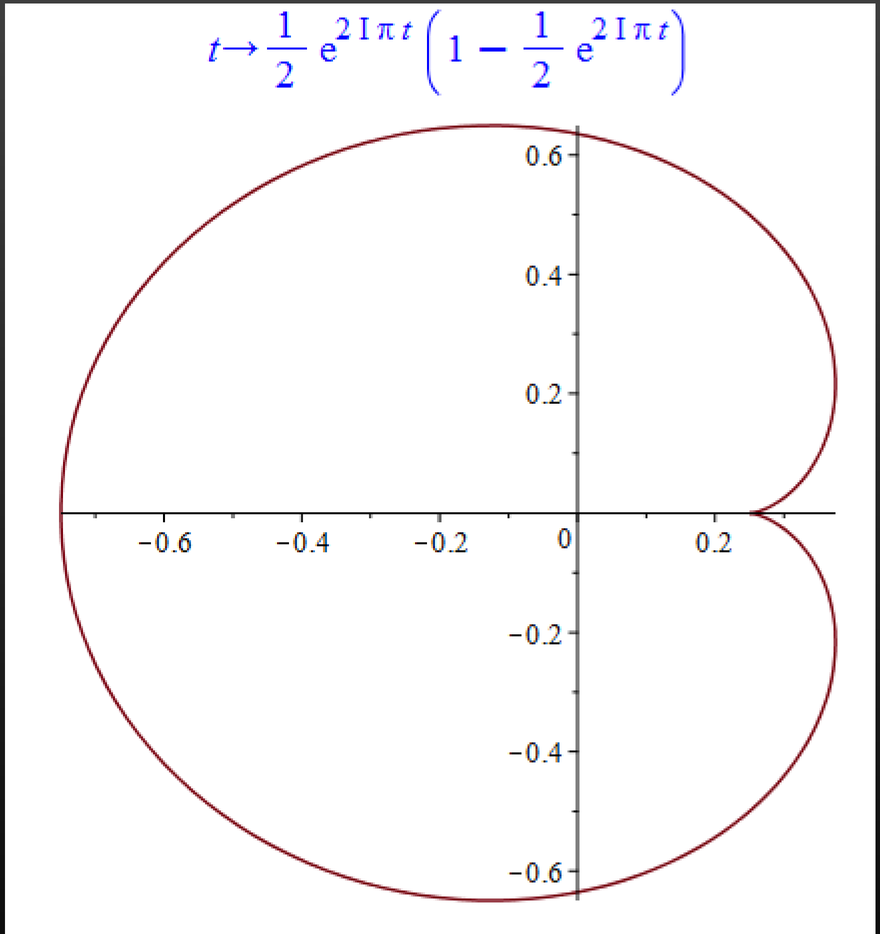


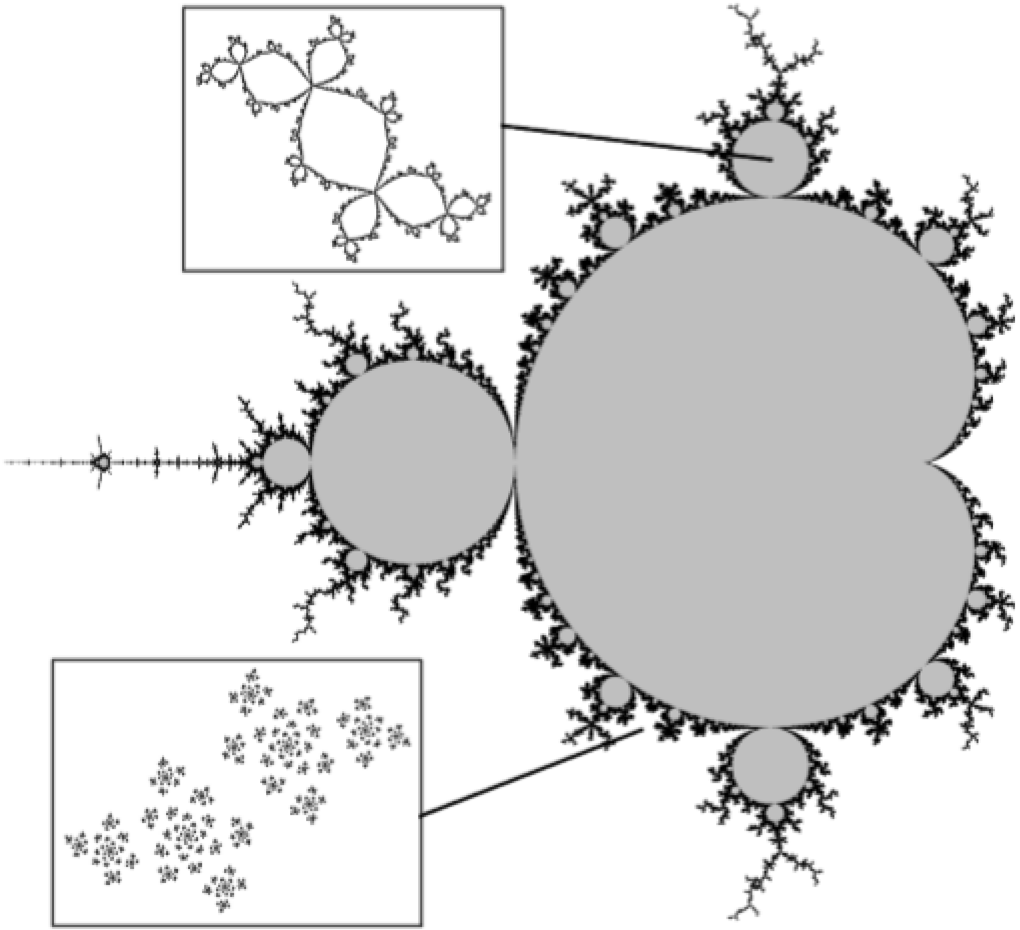
Main Body of M

Cardioid



$$C_{\frac{p}{q}} = \frac{e^{2\pi\frac{p}{q}i}}{2}\left(1-\frac{e^{2\pi\frac{p}{q}i}}{2}\right)$$





Dichotomy :

–$J(P_c)$ connected $\iff c \in M$

–$J(P_c)$ Cantor oterwise

The boundary ∂M is the bifurcation locus of the dynamics, i.e. the set of parameters c where the Julia set do not vary continuously with respect to c.

The slide is from “Polynomial Juliasets with positive measure” by XavierBuff&ArnaudCh´eritat Universit´ePaulSabatier(ToulouseIII) presented in memory of Adien Douday in

Theorem (Fatou Julia, 1919):
Julia set of $f: C \to C$ is

Either connected

or
Totally disconnected

Points c Outside of Mandelbrot set, Filled Julia set $z^2 + c$ is totally disconnected like dusts

Julia set of $z^2 + c$,

with c in Bulb of Mandelbrot set

Main Body of M

Cardioid has bulbs at $C_{\frac{p}{q}}$

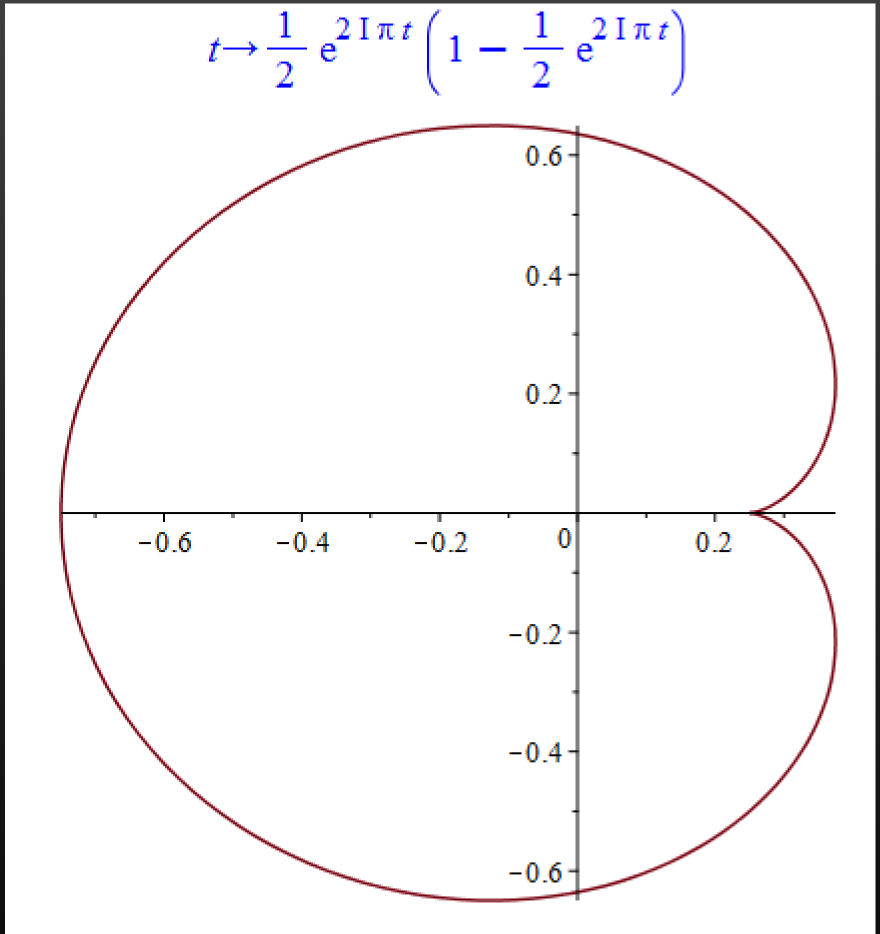
Julia set of $z^2 + C_{\frac{p}{q}}$

has

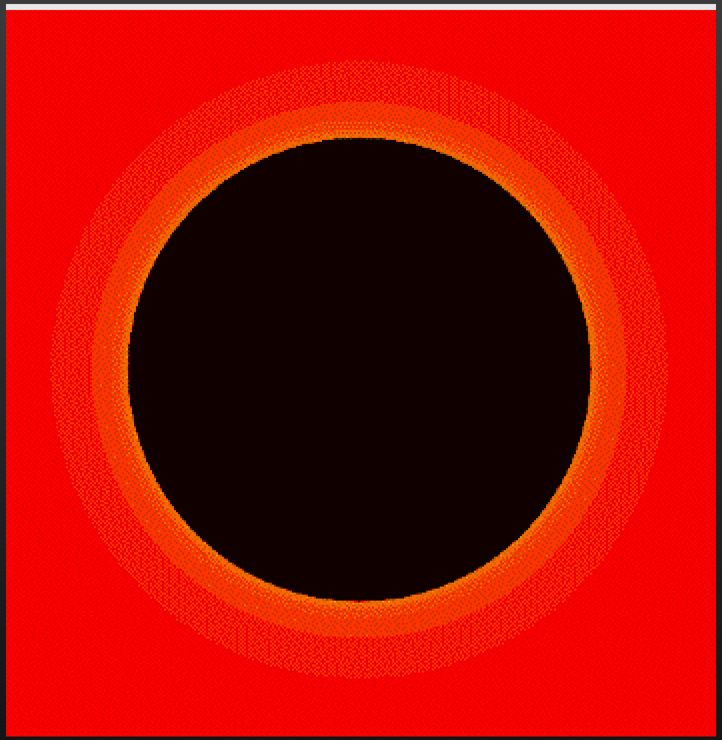
periodic points

with period q

$$C_{\frac{p}{q}} = \frac{e^{2\pi\frac{p}{q}i}}{2}\left(1 - \frac{e^{2\pi\frac{p}{q}i}}{2}\right)$$



$Julia\ set\ of\ z^2 + 0$ is the unit circle.



At golden ratio c

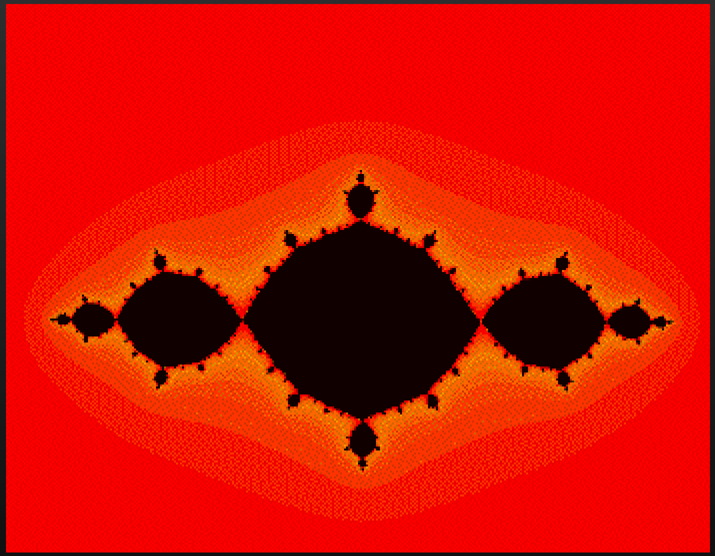
$$\frac{a+b}{a} = \frac{a}{b} = \phi$$

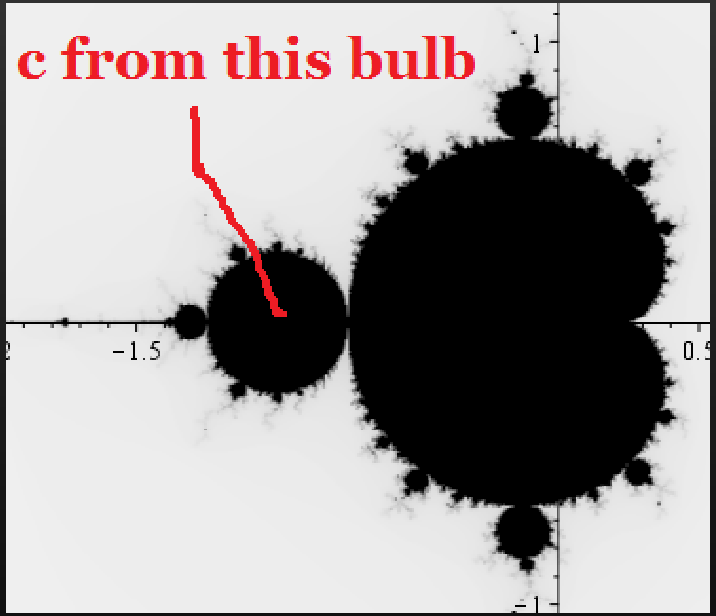
$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618 \ldots$$

Julia set of $z^2 + \phi$

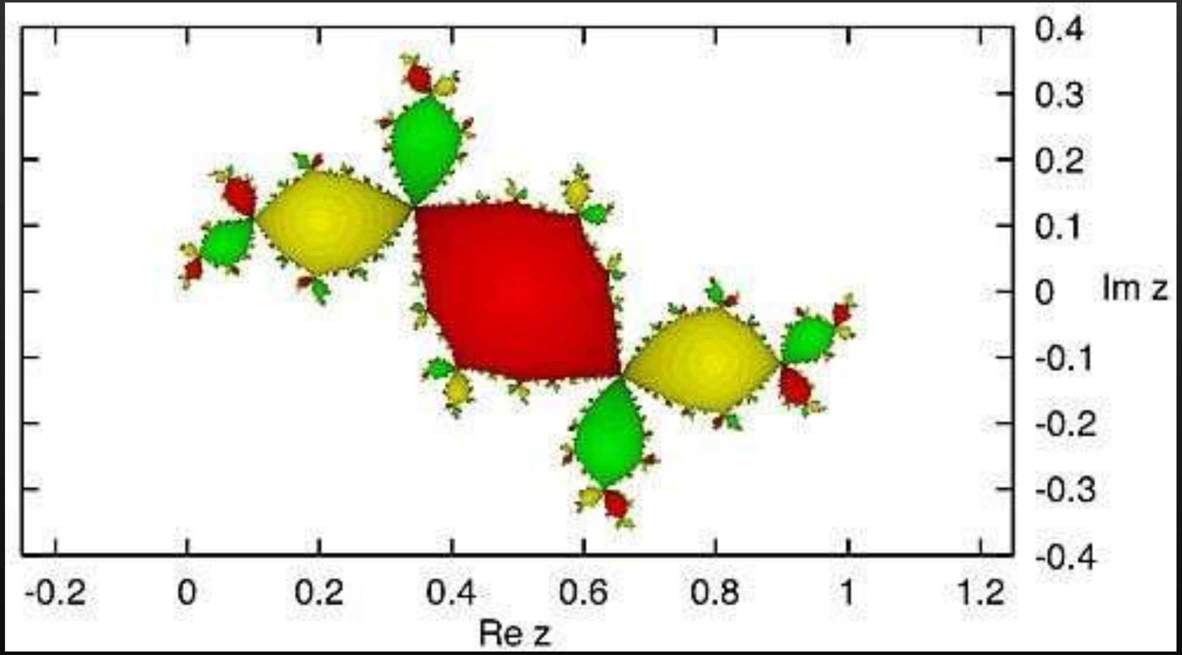
Julia set of $f(z) = z^2 - 1$

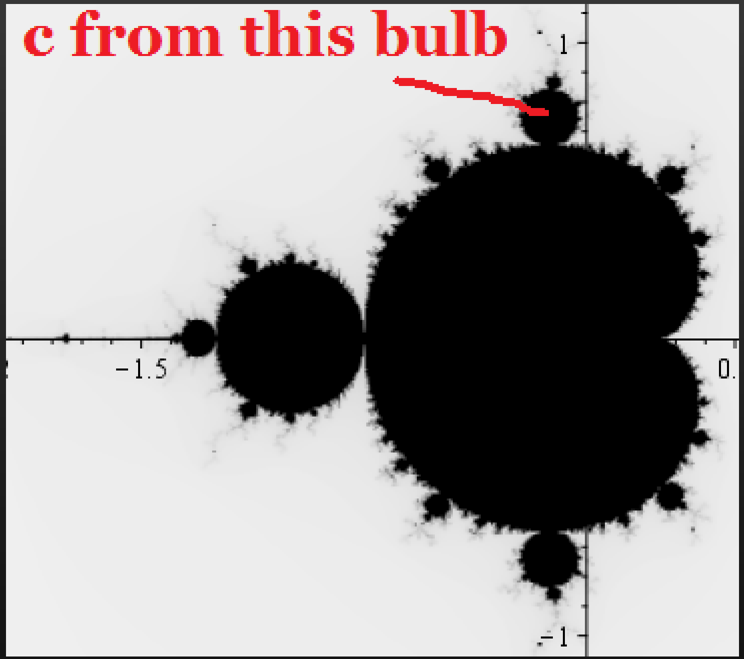
-1 is at the bulb C(1/2)





In the north bulb at 1/3 bulb period 3 Douady's rabbit Julia set of $z^2 + c$,
c=-0.125+ 0.731 I,
c~-0.12256+0.74486i

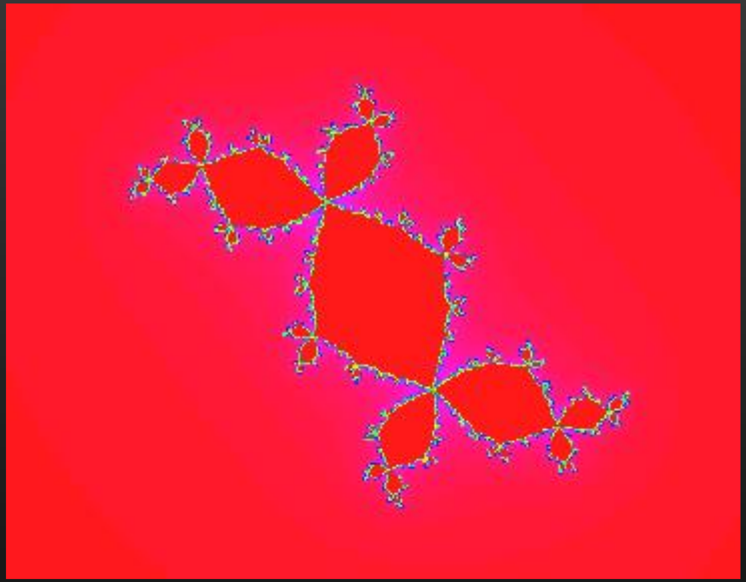




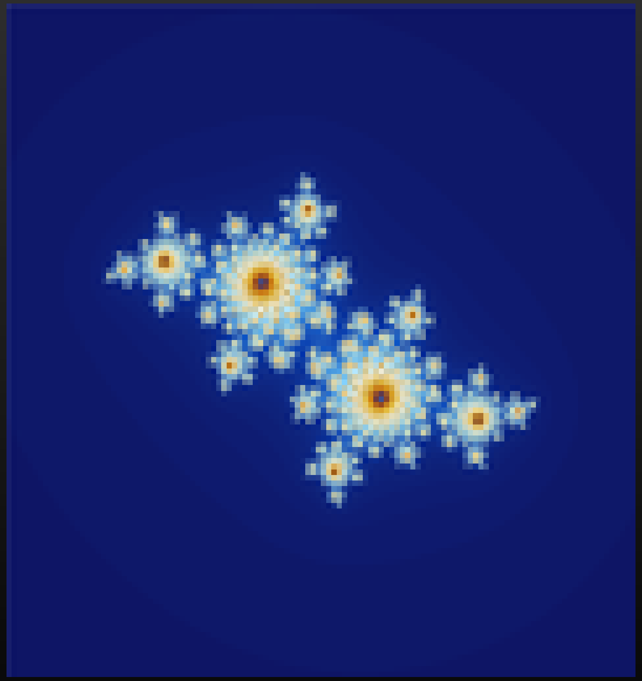
From Wikipedia

Maple produced Julia set Rabbits at c=-0.12+0.75I

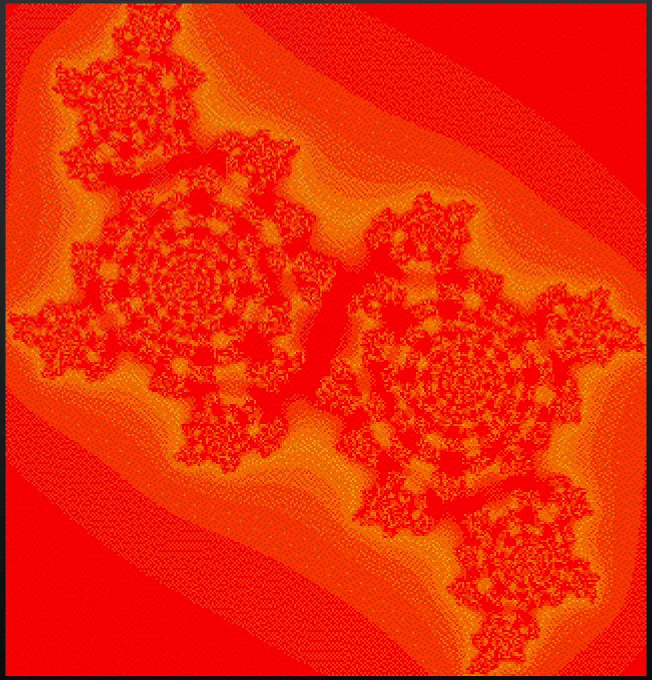
```
with(plots) :
JuliaSet := proc(X, Y)
 local Z, ct;
 Z := X + I* Y;
 for ct from 1 while ct < 120 and evalf(abs(Z)) < 4.0
   do
     Z := Z^2 -0.12 + 0.75 * I
   od;
  -ct;
end:
 densityplot('JuliaSet'(x, y),
  x=-2 ..2, y=-1.5 ..1.5,
  grid=[500, 500], colorstyle=HUE,
   scaling=constrained,
  style=PATCHNOGRID, axes=NONE);
```



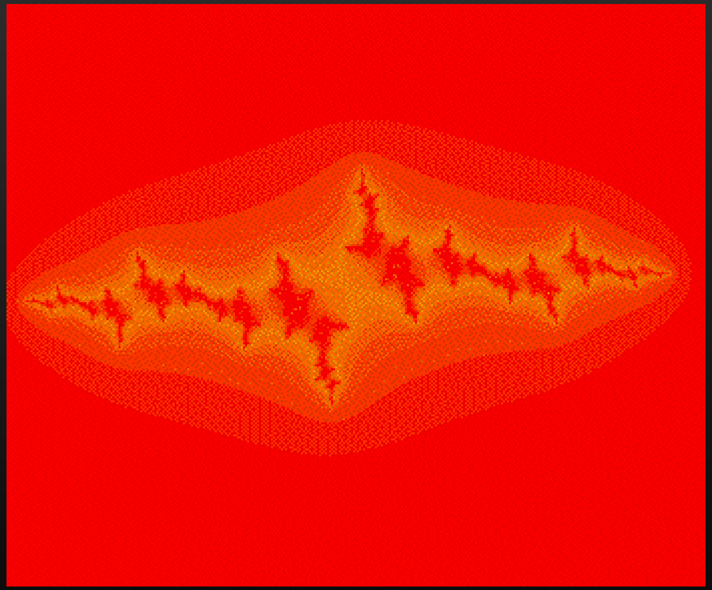
Julia set for f_c, $c=(\phi-2)+(\phi-1)i = -0.4+0.6i$



Getting chaotic



Chaos !!!!
Julia set for
filled Julia sets that is not connected is similar to the Cantor Middle Thirds Set.



Most Julia set we see have measure (area) 0.

Julia set of z^2*+c*
with c on the boundary of Mandelbrot set
are more *interesting points ,*
starting to be chaotic!!!

Cantor set: Totally disconnected like dusts

Recently, found Julia sets with postive measure.

More crazy/interesting points
Misiurewicz points

Misiurewicz points

Are Points c in M, for which

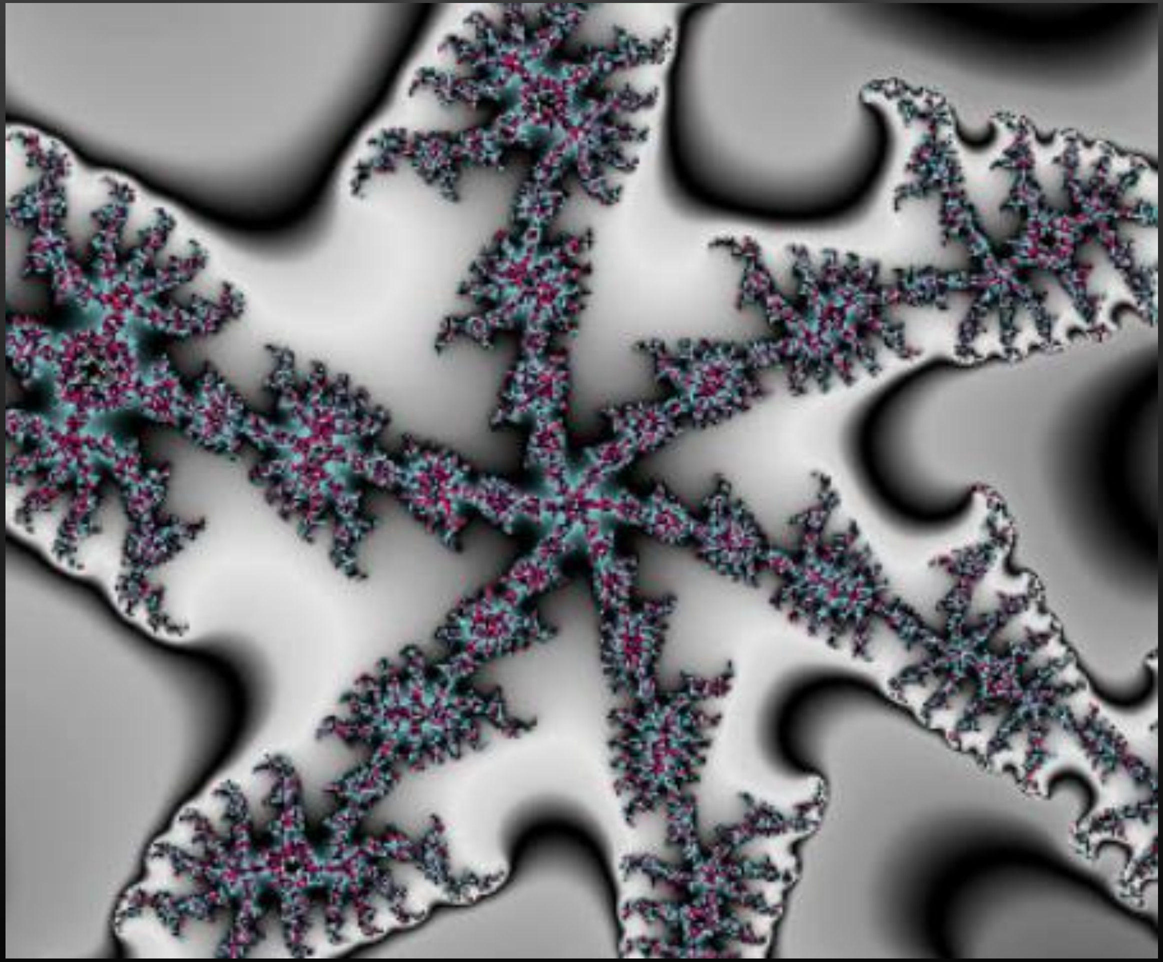
0 is preperiodic of $z^2 + c$.

Notice that 0 is the only critical point of $z^2 + c$.

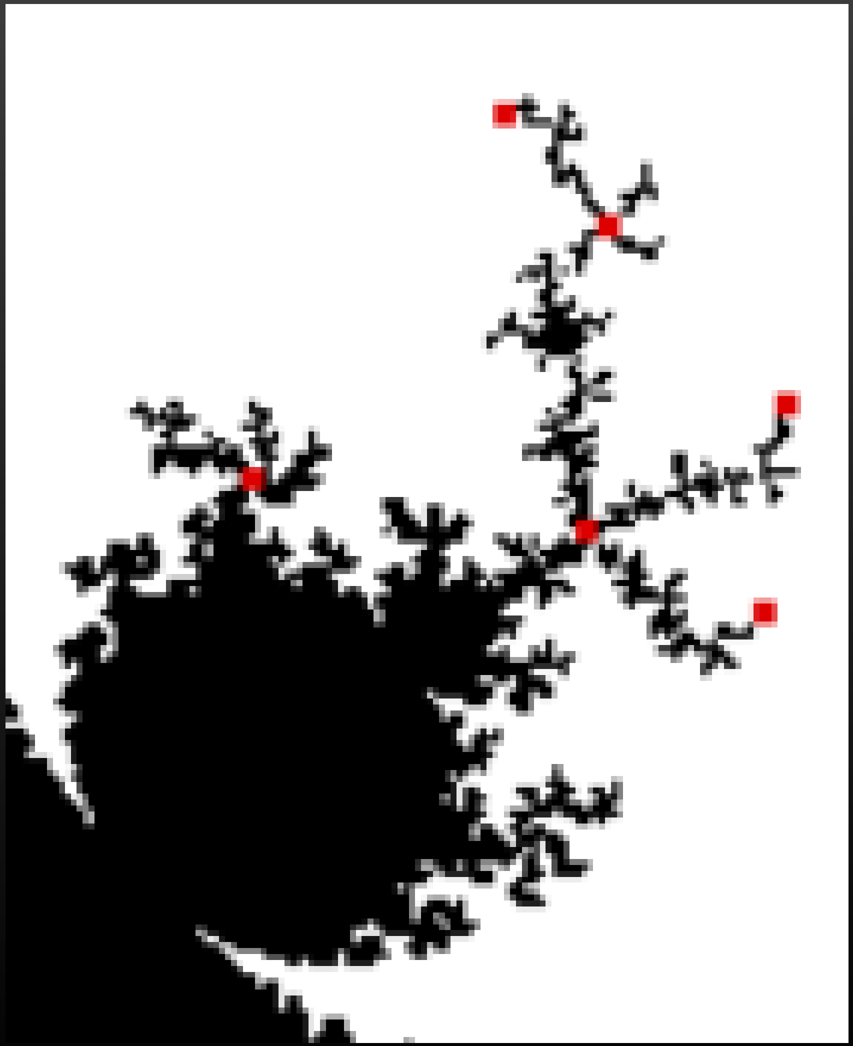
Misiurewicz points

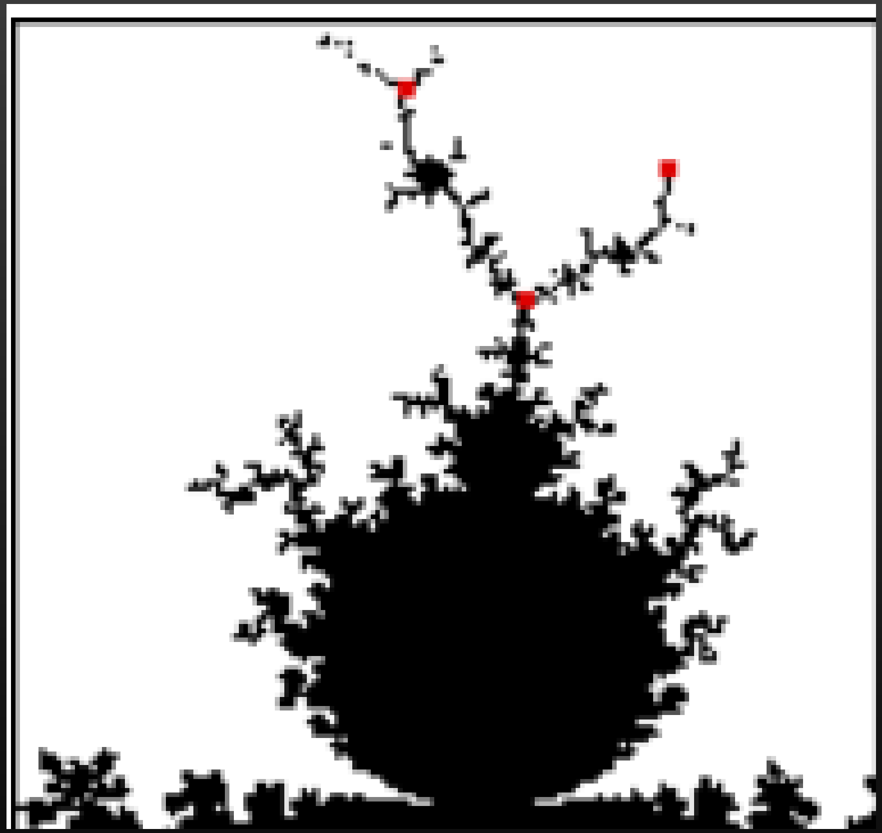
- Dense in the boundary of M
- Filled Julia set is equal to Julia set.
- Filled Julia set has no interior
- Mandelbrot set and Julia set are asymptotically similar.
- It disconnect M at least into 3 components

M-set: c= 0.4244 + 0.200759i



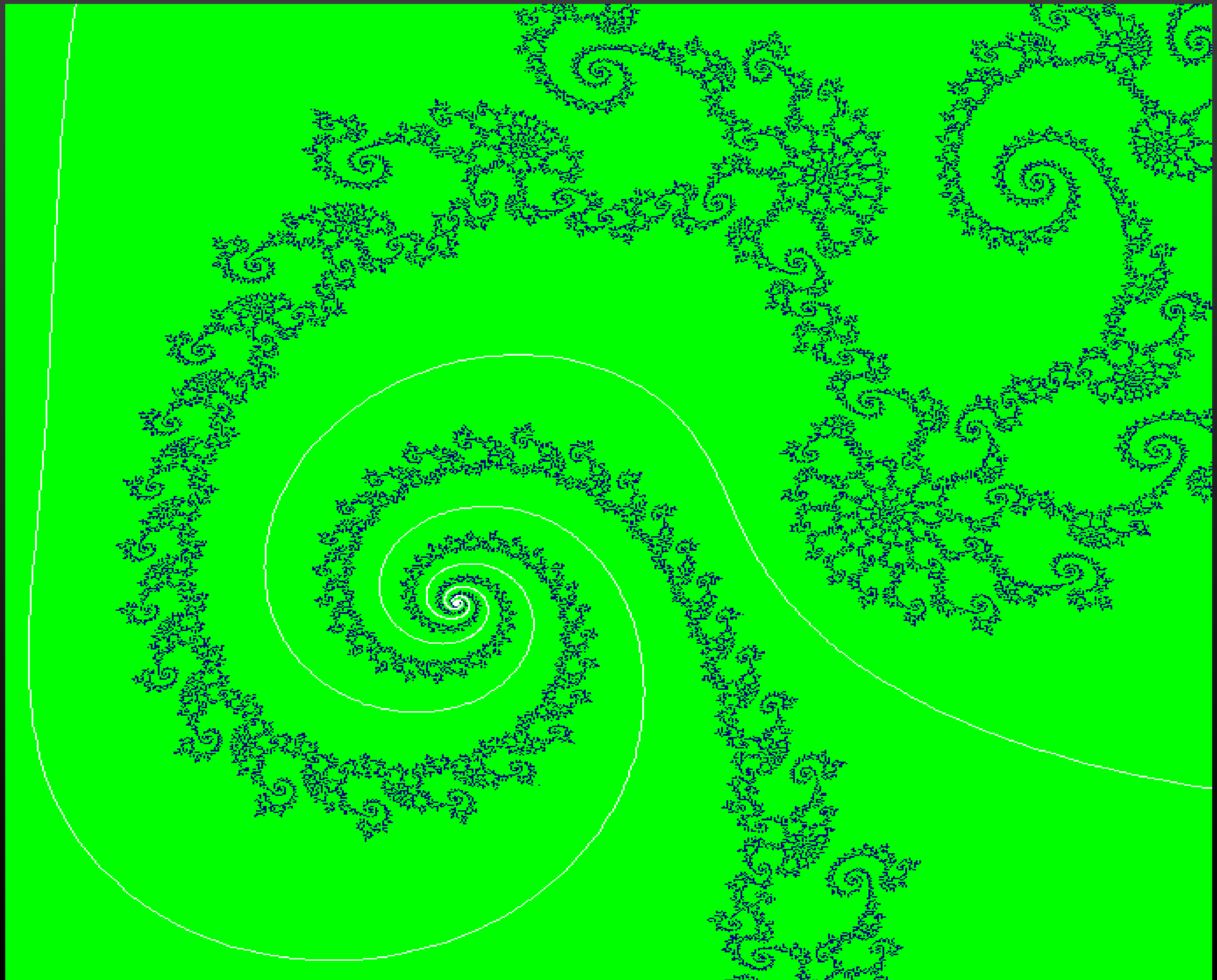
From Wikipdia
Misiurewicz points





$\mathrm{M}_{21} = -2,\ \mathrm{M}_{22} = \mathrm{i}, C = M_{23}$

https://en.wikipedia.org/wiki/Misiurewicz_point



Julia set of $f(z) = z^2 + 1$ *is totally disconnected*

Has a Lebesgue measure 0

OK to end here

Or go more

Hausdorff Dimension?

$$Partition\ of\ a\ set\ into\ N\ cells$$

with cells with diameter d

$$\delta\ such\ that$$

$$N\ (diameter)^{\delta} = 1$$

Example: Cantor set

Haussdorff dimension $\frac{\ln 2}{\ln 3}$

in n step

$$N = 2^n$$

$$2^n\left(\left(\frac{1}{3}\right)^n\right)^\delta = 1$$

$$\rightarrow \left(\frac{2}{3^\delta}\right)^n = 1 \rightarrow \delta = \frac{\ln 2}{\ln 3}$$

Example: Cantor set
Looks like dusts

Totally disconnected

Watch Movie

- Thank you